A Three-Dimensional Spectral Prediction Equation

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ABSTRACT

A generalization of the two-dimensional spectral forecast equations is suggested, whereby the atmospheric flow field in horizontal and vertical directions is represented in terms of orthogonal functions, which are eigenfunctions of certain differential operators in the three-dimensional equations. The technique is applied to the quasi-geostrophic potential vorticity equation.

The quasi-geostrophic potential vorticity is related to the stream function of the horizontal wind by a three-dimensional Laplacian in which the vertical derivative is modified by the standard static stability of the atmosphere. The orthogonal functions are chosen to be characteristic functions of this quasi-Laplacian operator for a given variation of static stability with height. The resulting spectral form of the potential vorticity equation is very similar to that of the barotropic vorticity equation.

1. Introduction

The complete numerical weather prediction equations cannot be solved exactly due to the nonlinear character of the equations. Thus, the meteorological literature is concerned with either solutions to the linearized equations or solutions by finite-difference techniques. Although the set of complete dynamical equations governing the motions of the atmosphere is now being solved by high-speed computers, there is no doubt that the large-scale motions of the atmosphere are described rather well by much simpler forms of the equations. In fact, when this was realized the modern developments in dynamic meteorology were initiated. Thus the historical studies of the stability properties of the linearized equations are based on some form of the quasi-geostrophic approximation which was up to recent years also a basic element of the numerical prediction models.

Although the nonlinear prediction equations are usually solved by finite-difference techniques in all three coordinate directions, this is not necessarily the best way. Time and again there have been suggestions in the literature proposing to replace the finite differences by one or another form of analytical function. In the early years of numerical forecasting, Eady (1952) and Eliassen (1952) introduced the two-parameter models in which the vertical layered structure of the atmosphere was rejected in favor of a representation in terms of given functions of the vertical coordinate. A few attempts have been made to extend this representation to more parameters but a satisfactory solution has not yet been proposed. On the other hand, useful horizontal representations of atmospheric variables in terms of orthogonal polynomials have been introduced in numerical meteorology by Silberman (1954) and later by Lorenz (1960) and Platzman (1960). The polynomials are chosen to be characteristic functions of the horizontal Laplace operator occurring in the quasi-geostrophic relationship between vorticity and height field.

The purpose of this paper is to extend the above ideas by representing the atmosphere in both horizontal and vertical directions in terms of orthogonal polynomials which are chosen in a logical manner. The technique is applied to the quasi-geostrophic potential vorticity equation which became very popular in early numerical prediction studies and recently has heating, the equations governing the motions in the atmosphere conserve the total vorticity and the total energy of the atmosphere. It is here required that the simplified equations are consistent in this respect.

The horizontal wind can be written as the sum of a nondivergent and an irrotational component

$$\mathbb{W} = \mathbb{k} \times \nabla\psi + \nabla\chi \tag{1}$$

where ψ is a stream function and χ a velocity potential, $\mathbb{k}$ is the vertical unit vector and ∇ is the horizontal gradient operator. As usual, pressure is used as the vertical coordinate. If ζ and D denote the vertical component of vorticity and the horizontal divergence, respectively, then from (1)

$$\zeta = \nabla^2\psi \qquad\qquad D = \nabla^2\chi \tag{2}$$

The horizontal equations of motion may be transformed into the vorticity equation and the divergence equation. A well-known simplified form of the divergence equation is the following quasi-geostrophic approximation

$$0 = -\nabla^2\Phi + f_0\nabla^2\psi \tag{3}$$

where f_0 is a constant value of the Coriolis parameter $f = 2\Omega \sin \phi$. If in addition the vorticity equation is truncated to the form

$$\frac{\partial\zeta}{\partial t} = -J(\psi, \zeta + f) - f_0 D \tag{4}$$

where J represents the Jacobian operator, then these equations satisfy the consistency requirements mentioned above.

The atmosphere is assumed to be always in quasi-hydrostatic balance and thus the vertical equation of motion reduces to

$$\frac{\partial\Phi}{\partial p} = -\alpha \tag{5}$$

values. This has an important consequence. All the terms in (9) are related to the height field according to (5) and (7). However, the standard value of the static stability is independent of the streamfield as defined by (10) while the first two terms of (9) will depend on the stream field only. This implies that the stream function and the static stability are effectively uncoupled.

Now let us define

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + (\mathbb{k} \times \nabla\psi).\nabla \tag{11}$$

Then from (2), (4), and (6) we obtain

$$\frac{d}{dt}(\nabla^2\psi + f) = f_0\frac{\partial\omega}{\partial p} \tag{12}$$

With (5), (7), and (10) we may write (9) as

$$\frac{d}{dt}(S\frac{\partial\psi}{\partial p}) = -\frac{\omega}{f_0} \tag{13}$$

where we defined

$$S(p) \equiv -\frac{P}{R}(\frac{\partial T}{\partial p} - \frac{\alpha}{c_p})_s^{-1} \tag{14}$$

Differentiating (13) with respect to pressure, noting (11), and adding (12), we find the prediction equation for the stream function

$$\frac{d}{dt}[\nabla^2\psi + f + f_0^2 \frac{\partial}{\partial p}(S\frac{\partial\psi}{\partial p})] = 0 \tag{15}$$

This is the quasi-geostrophic potential vorticity equation, which has become very popular in dynamic meteorology, and has been discussed recently by Phillips (1963).

The purpose of this paper is to find a vertical representation of the dependent variables ψ and ω which will satisfy the system of Eqs. (11) to (15) in a logical manner. In mathematical terms we seek the eigen-solutions of the vertical derivative-operator in (15) for given boundary conditions at the top and bottom of the model atmosphere. The usual assumption is that $\omega = 0$ at top and bottom. By (13) this imposes the to extrapolate (15) in time. We will, however, derive a series representation for the vertical motion which is at least consistent with the linearized forms of both (12) and (13). It is clear from (15) and (16) that the eigenfunctions are dependent on the vertical variation of $S(p)$. If S could be taken to be constant then ψ could be represented by a series of cosines and ω could be written as a series of sines. It will be shown in the following that a different distribution of S is more appropriate.

3. Standard Static Stability

The stability parameter S defined by (14) may be written in terms of the lapse rate of temperature $\gamma = -\partial T/\partial z$ by virtue of the hydrostatic equation

$$S \equiv - \frac{p}{R} \left(\frac{\partial T}{\partial p} - \frac{\alpha}{c_p}\right)^{-1} = \frac{gp^2}{R^2 T} (\gamma_d - \gamma)^{-1} \qquad (19)$$

It is seen that S is inversely proportional to the static stability $(\gamma_d - \gamma)$ and hence S will be called the static "instability". We will now consider the standard value of this static instability to be used in (15).

Observations show that the large-scale static stability $(\gamma_d - \gamma)$ tends to decrease with height in the middle troposphere, then increases sharply at tropopause level, and becomes nearly constant with height in the stratosphere. It may be seen immediately from (19) that the variation of the static instability S is quite different. Gates (1961) has measured the mean vertical distribution of various static stability parameters for January and July averaged over 45 United States radiosonde stations. The values of S obtained by averaging the January and July values are represented by the points in Fig. 1a. The variation with height is seen to be nearly linear up to a certain level which may be taken to represent the tropopause. Above the tropopause the static stability $(\gamma_d - \gamma)$ and the temperature are quasi-constant and thus from (19)

$$S = \frac{g}{R^2 T(\gamma_d - \gamma)} p^2 \simeq cp^2 \qquad p < p_t \qquad (20)$$

where p_t denotes the tropopause level.

Then from (21) $$S = S_o\ (1 - z) \tag{24}$$

By definition, the mathematical top is reached for $S = 0$, that is for $z = 1$. If p_1 denotes this upper boundary, then from (23)

$$p_1 = p_o - \frac{p_o - p_t}{q} \tag{25}$$

and also

$$z = \frac{p_o - p}{p_o - p_1} \tag{26}$$

Fig. 2a shows the variation with height of $\gamma_d - \gamma$ and the physical and mathematical upper boundary for $p_t = 175$ mb and $s_o = 85\ mb^2sec^2/m^2$. By definition, $\gamma_d - \gamma = 10°C/km$ for $p = p_t$ and thus from (19) we have $s_t = 1.7\ mb^2sec^2/m^2$ using the standard atmosphere temperature of 218°K in the stratosphere. The values of s_o, s_t, and p_t are based on Fig. 1a. From equation (22) then

$$q = \frac{85 - 1.7}{85} = .98$$

and hence from (25)

$$p_1 = 1000 - \frac{825}{.98} \simeq 160 \text{ mb}$$

It may be noted here that the model is completely determined by the values of s_o and p_t. It would seem that the tropopause level as suggested by Gates' data is rather high. Fig. 2b shows a model with the tropopause level closer to that of the standard atmosphere. The value of s_o is the same as in Fig. 2a but $p_t = 225$ mb and so from (19) with $\gamma_d - \gamma = 10°C/km$ we get $s_t = 2.8\ mb^2sec^2/m^2$. Then from (22) and (25)

$$q = .967 \qquad\qquad p_1 \simeq 200 \text{ mb}$$

The same figure also includes a more unstable configuration based on the value $s_o = 100\ mb^2sec^2/m^2$. For that case

$$q = .972 \qquad\qquad p_1 \simeq 200 \text{ mb}$$

then (30) and (31) become

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{1}{x}\frac{\partial \Phi}{\partial x} + \lambda^2 \Phi = 0 \tag{33}$$

$$x\frac{\partial \Phi}{\partial x} = 0 \qquad \text{for } x = 0,\ 2 \tag{34}$$

Equation (33) is Bessel's equation of order zero. The appropriate particular solution is Bessel's function of the first kind of order zero

$$\Phi = J_o(\lambda x) = \sum_{i=0}^{\infty} \frac{(-)^i}{(i!)^2}\left(\frac{\lambda x}{2}\right)^{2i} \tag{35}$$

This function is related to the Bessel function of order one as follows

$$\frac{dJ_o(x)}{dx} = -J_1(x) \tag{36}$$

Clearly, the boundary condition at x = 0 is satisfied. With (36) the remaining boundary condition becomes

$$J_1\ (2\lambda) = 0 \tag{37}$$

The eigenvalues are given by the infinite series of positive roots of (37). Thus $\lambda_m = \frac{1}{2} x_m$ if x_m are the zeros of the Bessel function of order one. The first eigenvalue $\lambda_o = 0$. The corresponding eigenfunctions are

$$\Phi_m\ (x) = J_o\ (\lambda_m x) \qquad m = 0, 1, 2, \ldots.. \tag{38}$$

The functions Φ_m are orthogonal on the interval (0,2) with weight function x. By using (33) to (38) one may derive

$$\int_0^2 x\Phi_m\ (x)\ \Phi_n\ (x)\ dx = \begin{cases} o & m \neq n \\ 2J_o^2\ (2\lambda_m) & m = n \end{cases} \tag{39}$$

then from (36) and (40)

$$(1 - z) \frac{dZ_m}{dz} = - \lambda_m X_m(z) \tag{46}$$

and from (44)

$$\lambda_m Z_m = \frac{dX_m}{dz} \tag{47}$$

The functions X_m are zero for $z = 1$ and also for $z = 0$ which follows from (37). It will be shown in the next section that the functions X_m are the required eigensolutions for the vertical motion. Here it may be noted that the following orthogonality relations hold. First, it can be shown readily, e.g., by partial integration of (41) and use of (45) and (47), that

$$\int_0^1 X_m(z)\, X_n(z) \frac{dz}{1-z} = \begin{cases} 0 & m \neq n \\ 1 & m = n \end{cases} \tag{48}$$

It follows from (41) and (47) that

$$\int_0^1 Z_m(z) \frac{dX_n}{dz}\, dz = \begin{cases} 0 & m \neq n \\ \lambda_m & m = n \end{cases} \tag{49}$$

and from (46) and (48) that

$$\int_0^1 \frac{dZ_m}{dz} X_n(z)\, dz = \begin{cases} 0 & m \neq n \\ -\lambda_m & m = n \end{cases} \tag{50}$$

The first few polynomials are shown in Fig. 4 plotted against a linear height scale.

5. Spectral Prediction Equation

The prediction equation will first be written in non-dimensional form. Defining $\mu \equiv \sin \phi$ where ϕ is latitude and taking Ω^{-1} as time unit, where Ω is the speed of the earth's rotation, and taking the radius of the earth as unit of length, we get $f \equiv 2\Omega \sin \phi = 2\mu$ and (27) becomes tions were checked by computing the orthogonality relationship (41). Table 2 shows the first few interaction coefficients. Clearly the interactions bear some resemblance to those of cosine functions, the major interactions occurring for $k = m \pm j$.

The computation of the nonlinear term in the spectral prediction equation may be somewhat simplified. Noting that $J_{mjk} = J_{mkj}$ and $I_{\gamma\beta\alpha} = -I_{\gamma\alpha\beta}$ and recalling the definition (57) we may write (58) as

$$(c_\gamma + \sigma d_m) \frac{d\psi_{\gamma m}}{dt} = 2i\ell_\gamma \psi_{\gamma m} + \frac{i}{2} \sum_\alpha \sum_\beta F_{\alpha\beta m} I_{\gamma\alpha\beta} \tag{61}$$

where

$$F_{\alpha\beta m} \equiv \sum_k \sum_j \psi_{\alpha k} \psi_{\beta j} [c_\alpha - c_\beta + \sigma(d_k - d_j)] J_{mkj} \tag{62}$$

Now $F_{\beta\alpha m} = -F_{\alpha\beta m}$ and so the product $F_{\alpha\beta m} I_{\gamma\alpha\beta}$ is symmetric in α and β. Further $I_{\gamma\alpha\beta} = 0$ for $\alpha = \beta$. Hence the double sum may be written

$$\frac{1}{2} \sum_\alpha \sum_\beta = \sum_\alpha \sum_{\beta=0}^{\alpha-1} \quad .$$

Due to the symmetry of J_{mkj}, the double sum in (62) may also be reduced to a summation of the form

$$\sum_k \sum_{j=0}^{k} \quad ;$$

note, however, that here the diagonal terms $(j = k)$ do not disappear.

Eq. (61) satisfies to predict the stream field. The vertical motion may then be computed from the diagnostic equations (12) or (13). In nondimensional form the vorticity equation becomes

$$\left(\frac{\partial}{\partial t} + \frac{\partial\psi}{\partial\lambda} \frac{\partial}{\partial\mu} - \frac{\partial\psi}{\partial\mu} \frac{\partial}{\partial\lambda} \right) \nabla^2\psi + 2 \frac{\partial\psi}{\partial\lambda} = \frac{-f_o}{p_o - p_1} \frac{\partial\omega}{\partial z} \tag{63}$$

and the adiabatic equation

$$\left(\frac{\partial}{\partial t} + \frac{\partial\psi}{\partial\lambda}\frac{\partial}{\partial\mu} - \frac{\partial\psi}{\partial\mu}\frac{\partial}{\partial\lambda} \right) \sigma(1-z) \frac{\partial\psi}{\partial z} = \frac{f_o}{p_o - p_1}\,\omega \tag{64}$$

Now let the vertical motion be represented by the series

$$\omega(\lambda, \mu, z, t) = \sum_{\gamma}\sum_{m} \omega_{\gamma m}(t)\, Y_{\gamma}(\lambda,\mu)\, X_m(z) \tag{65}$$

where $X_m(z)$ is given by (45) and hence is zero for z=0 and z=1, thus satisfying the vertical boundary conditions. Note also that $X_o(z) = 0$ according to (45) since $\lambda_o = 0$. Substituting the expansions (55) and (65) into (63), applying the orthogonality relationships (54), (41), and in particular (49), we obtain

$$c_{\gamma}\frac{d\psi_{\gamma m}}{dt} - 2i\ell_{\gamma}\psi_{\gamma m} - \frac{i}{2}\sum_{\alpha}\sum_{\beta} G_{\alpha\beta m}\, I_{\gamma\alpha\beta} = \frac{f_o\lambda_m}{p_o - p_1}\,\omega_{\gamma m} \tag{66}$$

where

$$G_{\alpha\beta m} \equiv (c_{\alpha} - c_{\beta}) \sum_{k}\sum_{j} \psi_{\alpha k}\, \psi_{\beta j}\, J_{mkj} \tag{67}$$

For completeness we may derive the spectral form of (64). Using the orthogonality relations (48) and (50) we obtain

$$\sigma d_m \frac{d\psi_{\gamma m}}{dt} - \frac{i}{2}\sum_{\alpha}\sum_{\beta} H_{\alpha\beta m}\, I_{\gamma\alpha\beta} = -\frac{f_o\lambda_m}{p_o - p_1}\,\omega_{\gamma m} \tag{68}$$

where

$$H_{\alpha\beta m} \equiv \sigma \sum_{k}\sum_{j} \psi_{\alpha k}\, \psi_{\beta j}\, (d_k - d_j)\, J_{mkj} \tag{69}$$

where the function $H_{\alpha\beta m}$ is obtained by a partial integration after applying (46). Eqs. (68) and (69) may, of course, be obtained directly

For actual computations the series will be truncated to a finite set. However, following Platzman (1960) we can show that (74) is still satisfied if all components of the set are included in the double sum in (61) and (62). Thus the relationship (74) is used as a check on time-truncation errors.

6. Conclusion

Three-dimensional atmospheric flow fields may be represented in terms of orthogonal polynomials which are characteristic functions for certain differential operators occurring in the forecast equations. In the present paper the orthogonal functions are chosen to be eigenfunctions of the three-dimensional quasi-Laplacian operator in the potential vorticity equation. Thus the functions representing the vertical variation of the flow field are determined by the variation of the standard static stability with height. A simple relationship between the stability parameter and pressure is suggested by observations and has been used in this paper. Another set of eigenfunctions might be found for a different variation of static stability with height. However, the approximation used is thought to be satisfactory for describing the large-scale atmospheric motions. Moreover, a more accurate description of the standard static stability is not justified in view of the approximations made in deriving the quasi-geostrophic potential vorticity equation.

The spectral form of the potential vorticity equation is presently being used in a study of barotropic and baroclinic instability in the atmosphere. The main objectives of that study are: (1) comparison of linear and nonlinear instability, (2) evaluation of the influence of the initial form of the perturbation on its growth-rate, and (3) the relation between stability characteristics and truncation of the series representation for the flow field with respect to the horizontal and the vertical. The spectral prediction equation derived in this paper is particularly well-suited for this kind of study.

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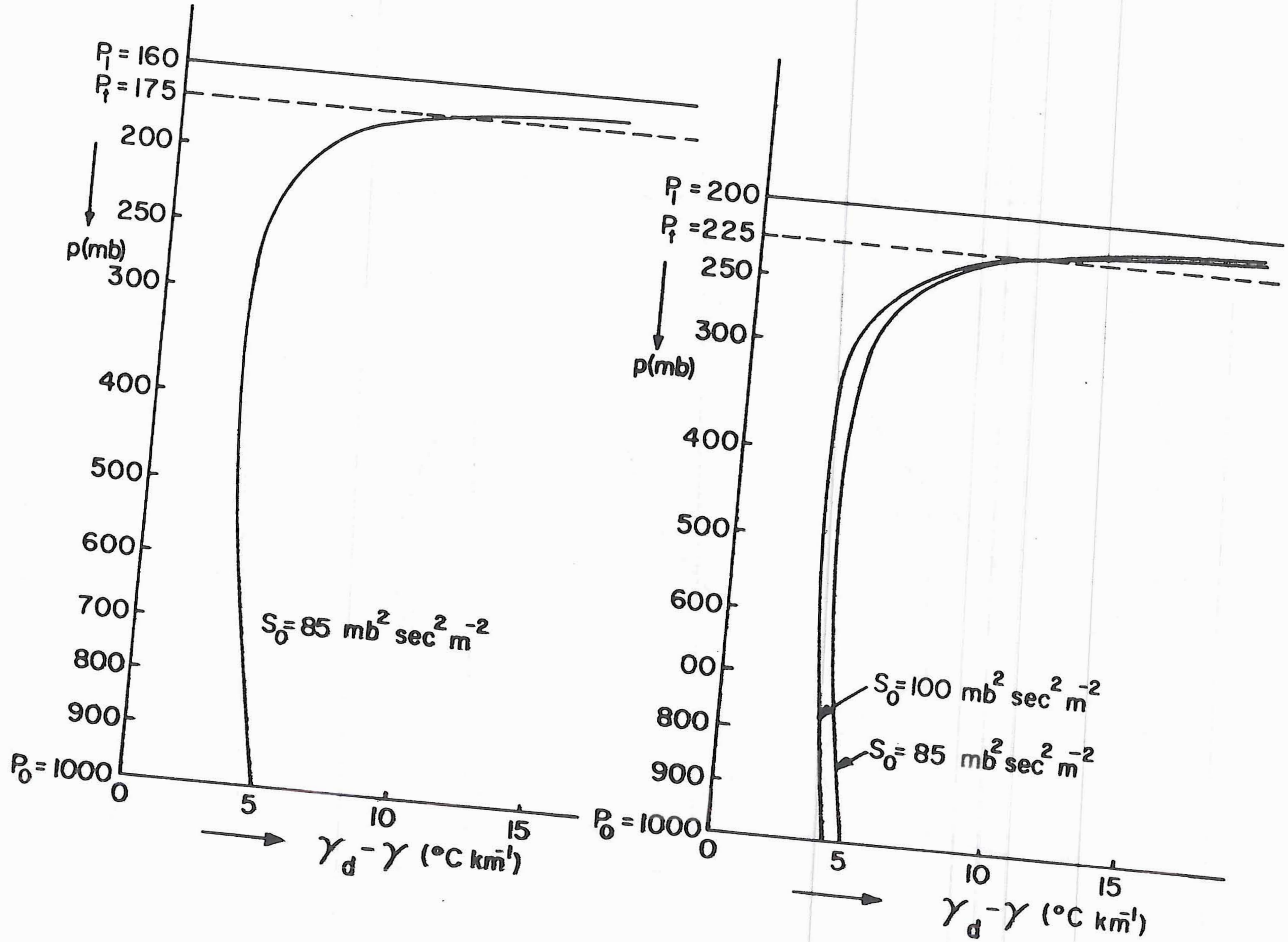

Fig. 2 Variation of static stability with height in model atmosphere for various values of parameters s_o and p_t

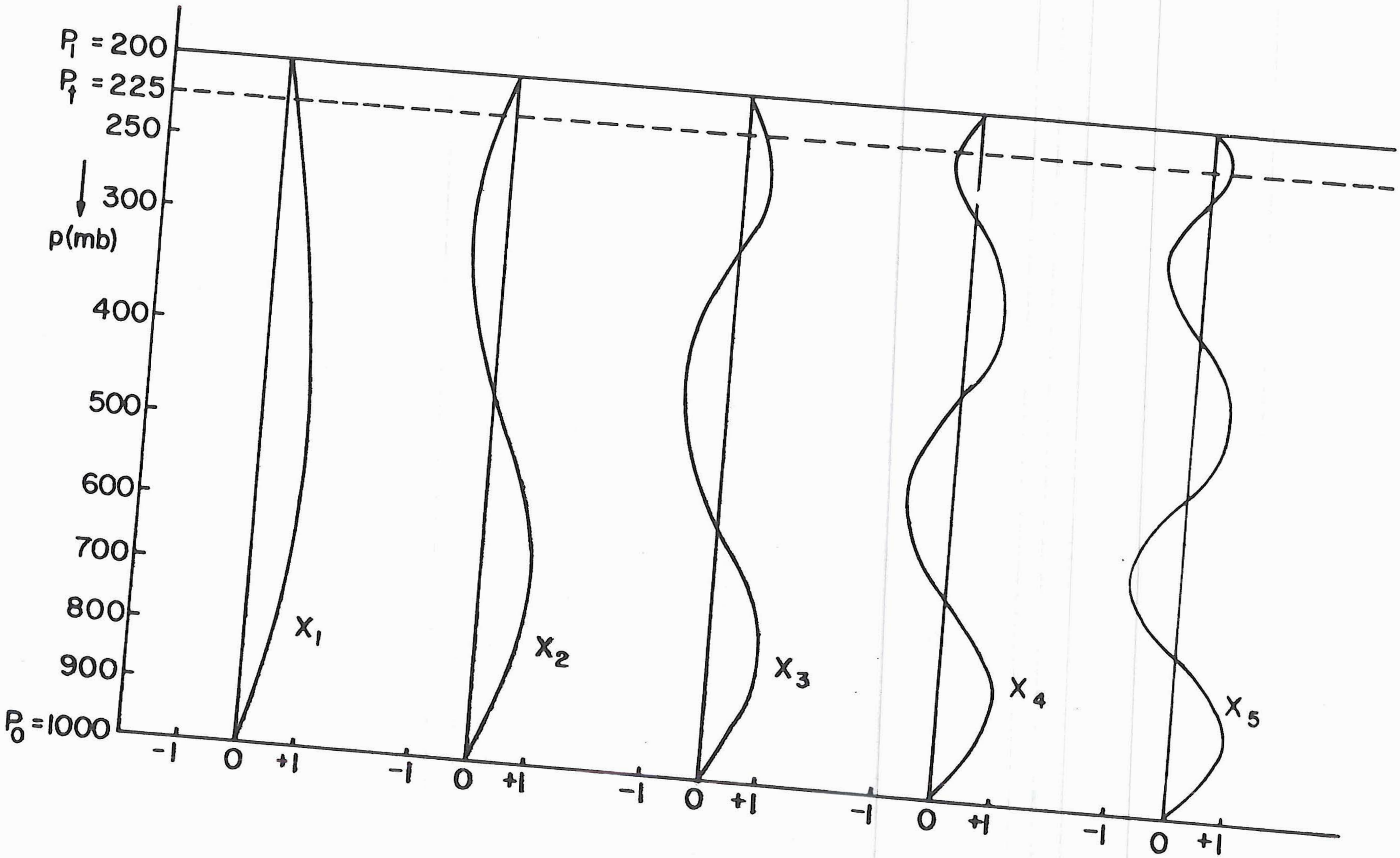

Fig. 4 Polynomials $X_m \equiv -\alpha_m \sqrt{1-z}\, J_1\, (2\lambda_m \sqrt{1-z}\,)$

		j=0	j=1	j=2	j=3	j=4
m=0	k=0	1.	0.	0.	0.	0.
	k=1	0.	1.	0.	0.	0.
	k=2	0.	0.	1.	0.	0.
	k=3	0.	0.	0.	1.	0.
	k=4	0.	0.	0.	0.	1.
m=1	k=0	0.	1.	0.	0.	0.
	k=1	1.	-.874663428461	.886783279415	-.010591634078	-.001139174754
	k=2	0.	.886783279415	-.749717066387	.878610435014	-.013381827554
	k=3	0.	-.010591634078	.878610435014	-.727026943622	.876164779497
	k=4	0.	-.001139174754	-.013381827554	.876164779497	-.718838439866
m=2	k=0	0.	0.	1.	0.	0.
	k=1	0.	.886783279415	-.749717066387	.878610435014	-.013381827554
	k=2	1.	-.749717066387	.517959559900	-.615062150507	.867803973740
	k=3	0.	.878610435014	-.615062150507	.482675398311	-.587059840133
	k=4	0.	-.013381827554	.867803973740	-.587059840133	.470902423164
m=3	k=0	0.	0.	0.	1.	0.
	k=1	0.	-.010591634078	.878610435014	-.727026943622	.876164779497
	k=2	0.	.878610435014	-.615062150507	.482675398311	-.587059840133
	k=3	1.	-.727026943622	.482675398311	-.472960848340	.439708915718
	k=4	0.	.876164779497	-.587059840133	.439708915718	-.439766904532
m=4	k=0	0.	0.	0.	0.	1.
	k=1	0.	-.001139174754	-.013381827554	.876164779497	-.718838439866
	k=2	0.	-.013381827554	.867803973740	-.587059840133	.470902423164
	k=3	0.	.876164779497	-.587059840133	.439708915718	-.439766904532
	k=4	1.	-.718838439866	.470902423164	-.439766904532	.387851034112

Table 2: Interaction coefficients J_{mkj}