Design of an Axisymmetric Primitive Equation Tropical Cyclone Model

By
James J. Hack and Wayne H. Schubert

Department of Atmospheric Science
Colorado State University
Fort Collins, Colorado

The research reported here has been supported by the Global Atmospheric Research Program, Climate Dynamics Research Section, National Science Foundation, Grants DES 74-11438, ATM 76-09370 and OCD 74-21678.
December 1976

Colorado State University

Department of Atmospheric Science

Paper No. 263

DESIGN OF AN AXISYMMETRIC PRIMITIVE EQUATION TROPICAL CYCLONE MODEL

James J. Hack
Wayne H. Schubert

The research reported here has been supported by the Global Atmospheric Research Program, Climate Dynamics Research Section, National Science Foundation, Grants DES 74-11438, ATM 76-09370 and OCD 74-21678.

Department of Atmospheric Science
Colorado State University
Fort Collins, Colorado

December 1976

Atmospheric Science Paper No. 263

CONTENTS

ABSTRACT

An axisymmetric primitive equation tropical cyclone model in the sigma coordinate is presented. The parameterization of the convective scale vertical transports of heat, moisture and momentum follow the theory presented by Arakawa and Schubert. The equations of the theory are grouped into three parts: feedback, static control and dynamic control. The dynamic control is formulated in terms of a linear programming problem which is solved by the simplex method. The space differencing schemes used in the model follow those developed by Arakawa and are designed such that the discrete analogues of certain integral constraints are maintained.

1.0 INTRODUCTION

A tropical cyclone generally consists of motions on two widely different space and time scales. The large-scale motion is on a space scale of hundreds of kilometers and a time scale of days and consists of cyclonic inflow in the lower troposphere and anticyclonic outflow in the upper troposphere. The small-scale motion is on a space scale of kilometers and a time scale of tens of minutes and consists of the many cumulus and cumulonimbus clouds which are organized by the large-scale motion. The cloud field may be organized in such a way as to give rise to a heat source which causes amplification of the large-scale disturbance, which in turn amplifies the cloud field. A mathematical theory of this positive feedback mechanism was first proposed by Charney and Eliassen (1964) and Ooyama (1964). Both Charney and Eliassen and Ooyama modeled the large-scale or cyclone scale explicitly but treated the small-scale or cumulus scale only implicitly, i.e. by what is now commonly referred to as cumulus parameterization. Although Charney and Eliassen's and Ooyama's work dealt only with the initial growth of a tropical depression, their approach stimulated efforts to numerically simulate the life cycle of tropical cyclones with axisymmetric models using the gradient balance assumption (Ogura, 1964; Kuo, 1965; Ooyama, 1969 a, b; Sundqvist, 1970 a, b, 1972; Peng and Kuo, 1975), with axisymmetric models using the primitive equations (Yamasaki, 1968 a, b, c; Rosenthal, 1969 a, b, 1970, 1971; Kurihara, 1975), and with fully three dimensional models using the primitive equations (Anthes et al., 1971 a, b; Anthes, 1972; Mathur, 1972; Kurihara and Tuleya, 1974; Tuleya and

Kurihara, 1975; Madala and Piacsek, 1975). The generally accepted criticisms of all the above works are the methods by which cumulus clouds are implicitly incorporated into the models, i.e. the methods by which cumulus clouds are parameterized. Recently, however, considerable advance in cumulus parameterization theory has been made by Ooyama (1971) and Arakawa and Schubert (1974). The theory of Ooyama (1971) was tested in a tropical cyclone model by Rosenthal (1973). The purpose of this paper is to present the design of a tropical cyclone model which includes the parameterization theory of Arakawa and Schubert (1974). Work along lines similar to ours is also being performed by Rosenthal.

2.0 GOVERNING EQUATIONS

2.1 Sigma coordinate

As a vertical coordinate we shall use the sigma coordinate given by Arakawa (1972) as

$$\sigma \equiv \frac{p - p_T}{p_S - p_T} \equiv \frac{p - p_T}{\pi} \quad , \tag{2.1}$$

where the top boundary pressure p_T is a specified constant and the surface pressure p_S (or equivalently π) is a function of the horizontal coordinates and time. The upper and lower boundaries are respectively $\sigma = 0$ and $\sigma = 1$. In the special case where $p_T = 0$, (2.1) reduces to the original definition introduced by Phillips (1957).

2.2 Hydrostatic equation

With pressure as the vertical coordinate, the hydrostatic equation is written

$$\frac{\partial \Phi}{\partial p} = -\alpha \quad , \tag{2.2}$$

where $\Phi \equiv gz$. From (2.1) and (2.2) we can easily show that

$$\frac{\partial \Phi}{\partial \sigma} = -\pi\alpha \quad , \tag{2.3}$$

which is the hydrostatic equation in the σ-coordinate. Since (2.3) can also be written

$$\rho dxdydz = -\frac{\pi}{g} dxdyd\sigma \quad , \tag{2.4}$$

the quantity $\frac{\pi}{g}$ can be thought of as a "density", i.e. the mass per unit "volume" of $xy\sigma$ space, or the mass of a vertical column with unit horizontal cross section. Although constant in the vertical, $\frac{\pi}{g}$ may vary with the horizontal coordinates and time.

2.3 Equation of continuity

In pressure coordinates the equation of continuity is

$$\nabla_p \cdot \mathbb{V} + \frac{\partial \omega}{\partial p} = 0 \ , \tag{2.5}$$

where ω is the vertical p velocity $\frac{dp}{dt}$. It can be shown that the del operator on a constant pressure surface is related to the del operator on a constant sigma surface by

$$\nabla_p = \nabla_\sigma - \frac{\sigma}{\pi} \nabla \pi \frac{\partial}{\partial \sigma} \ . \tag{2.6}$$

Using (2.1) and (2.6), (2.5) can be written

$$\frac{\partial \pi}{\partial t} + \nabla_\sigma \cdot (\pi \mathbb{V}) + \frac{\partial}{\partial \sigma} (\pi \dot{\sigma}) = 0 \ , \tag{2.7}$$

where $\dot{\sigma}$ is the vertical σ velocity $\frac{d\sigma}{dt}$. In our tropical cyclone model we make use of cylindrical coordinates and of the axisymmetric assumption. The continuity equation (2.7) can then be written

$$\frac{\partial \pi}{\partial t} + \frac{\partial}{r \partial r} (\pi r u) + \frac{\partial}{\partial \sigma} (\pi \dot{\sigma}) = 0 \ . \tag{2.8}$$

As upper and lower boundary conditions we require that air particles do not cross the $\sigma = 0$ and $\sigma = 1$ levels, i.e.

$$\dot{\sigma} = 0 \text{ at } \sigma = 0, 1 \ . \tag{2.9}$$

Integrating (2.8) over the entire vertical column and using the boundary conditions (2.9), we obtain

$$\frac{\partial\pi}{\partial t} = -\frac{\partial}{r\partial r}\int_0^1 \pi ru\,d\sigma \; . \qquad (2.10)$$

Integrating (2.8) from the top of the vertical column to σ and using the upper boundary condition, we obtain

$$\pi\dot{\sigma} = -\left[\sigma\frac{\partial\pi}{\partial t} + \frac{\partial}{r\partial r}\int_0^\sigma \pi ru\,d\sigma'\right] . \qquad (2.11)$$

Thus, knowing the radial wind component u, (2.10) can be used to compute $\frac{\partial\pi}{\partial t}$, then (2.11) can be used to diagnose $\pi\dot{\sigma}$ at any σ level.

With the assumption of axisymmetry, the individual time derivative of an arbitrary scalar quantity ψ is given by

$$\frac{d\psi}{dt} = \frac{\partial\psi}{\partial t} + u\frac{\partial\psi}{\partial r} + \dot{\sigma}\frac{\partial\psi}{\partial\sigma} \; . \qquad (2.12)$$

Using the continuity equation (2.8), we obtain the flux form of the individual time derivative of an arbitrary scalar quantity ψ as

$$\pi\frac{d\psi}{dt} = \frac{\partial}{\partial t}(\pi\psi) + \frac{\partial}{r\partial r}(\pi ru\psi) + \frac{\partial}{\partial\sigma}(\pi\dot{\sigma}\psi) \; . \qquad (2.13)$$

2.4 Equations of motion

The horizontal component of the vector equation of motion in pressure coordinates is

$$\frac{d\mathbb{V}}{dt} + f\,\mathbb{k}\times\mathbb{V} + \nabla_p\Phi = \mathbb{F} \; , \qquad (2.14)$$

where $\frac{d\mathbb{V}}{dt}$ is the horizontal acceleration, $f\,\mathbb{k}\times\mathbb{V}$ the horizontal coriolis acceleration, $\nabla_p \Phi$ the horizontal pressure gradient force per unit mass, and $\mathbb{F}$ the horizontal frictional force per unit mass. Using (2.3) and (2.6) we can rewrite (2.14) as

$$\frac{d\,\mathbb{V}}{dt} + f\,\mathbb{k} \times \mathbb{V} + \nabla_\sigma \Phi + \sigma\alpha\nabla\pi = \mathbb{F} \ . \qquad (2.15)$$

Equation (2.15) can also be written as

$$\frac{\partial\,\mathbb{V}}{\partial t} + \dot{\sigma}\,\frac{\partial\,\mathbb{V}}{\partial\sigma} + (f + \zeta)\,\mathbb{k} \times \mathbb{V} + \nabla_\sigma\left(\frac{\mathbb{V}^2}{2} + \Phi\right) + \sigma\alpha\nabla\pi = \mathbb{F} \ , \qquad (2.16)$$

where ζ is the vertical component of the relative vorticity, i.e. $\mathbb{k} \cdot \nabla \times \mathbb{V}$. In the special case of axisymmetry the relative vorticity is given by

$$\zeta = \frac{\partial}{r\partial r}\,(rv). \qquad (2.17)$$

Using (2.17) we can write the radial component of (2.16) as

$$\frac{\partial u}{\partial t} + \dot{\sigma}\,\frac{\partial u}{\partial \sigma} - \left[f + \frac{\partial}{r\partial r}\,(rv)\right]v$$

$$+ \frac{\partial}{\partial r}\,\left(\frac{u^2}{2} + \frac{v^2}{2} + \Phi\right) + \sigma\alpha\,\frac{\partial\pi}{\partial r} = F_r \qquad (2.18)$$

and the tangential component as

$$\frac{\partial v}{\partial t} + \dot{\sigma}\,\frac{\partial u}{\partial \sigma} + \left[f + \frac{\partial}{r\partial r}\,(rv)\right]u = F_\phi \ . \qquad (2.19)$$

Equations (2.18) and (2.19) can be combined with the continuity equation (2.8) to yield the flux form of the radial momentum equation

$$\frac{\partial}{\partial t}\,(\pi u) + \frac{\partial}{r\partial r}\,(\pi r u u) + \frac{\partial}{\partial\sigma}\,(\pi\dot{\sigma}u) - \left(f + \frac{v}{r}\right)\pi v$$

$$+ \pi\left(\frac{\partial\Phi}{\partial r} + \sigma\alpha\,\frac{\partial\pi}{\partial r}\right) = \pi F_r \ , \qquad (2.20)$$

and the flux form of the tangential momentum equation

$$\frac{\partial}{\partial t}(\pi v) + \frac{\partial}{r\partial r}(\pi r u v) + \frac{\partial}{\partial \sigma}(\pi \dot{\sigma} v) + (f + \frac{v}{r})\pi u = \pi F_{\phi} \ . \tag{2.21}$$

2.5 Equation of state

The equation of state is given by

$$\alpha = \frac{RT}{p} \ , \tag{2.22}$$

where R is the gas constant for dry air.

2.6 First law of thermodynamics

The first law of thermodynamics can be written as

$$\frac{d}{dt}(c_p T) = \omega\alpha + Q \ , \tag{2.23}$$

or in flux form as

$$\frac{\partial}{\partial t}(\pi c_p T) + \nabla_{\sigma}\cdot(\pi \mathbb{V} c_p T) + \frac{\partial}{\partial \sigma}(\pi \dot{\sigma} c_p T) = \pi\omega\alpha + \pi Q \ . \tag{2.24}$$

Defining

$$\theta = T\left(\frac{p_o}{p}\right)^{\kappa} \ , \tag{2.25}$$

and using (2.1), we can rewrite (2.24) as

$$\frac{\partial}{\partial t}(\pi c_p T) + \nabla_{\sigma}\cdot(\pi \mathbb{V} c_p T) + \left(\frac{p}{p_o}\right)^{\kappa} \frac{\partial}{\partial \sigma}(\pi \dot{\sigma} c_p \theta)$$

$$= \pi\sigma\alpha(\frac{\partial}{\partial t} + \mathbb{V}\cdot\nabla)\pi + \pi Q \ . \tag{2.26}$$

In cylindrical coordinates and with the axisymmetric assumption, (2.26) can be written

$$\frac{\partial}{\partial t}(\pi c_p T) + \frac{\partial}{r\partial r}(\pi r u c_p T) + \left(\frac{p}{p_o}\right)^{\kappa} \frac{\partial}{\partial \sigma}(\pi \dot{\sigma} c_p \theta)$$

$$= \pi\sigma\alpha\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial r}\right)\pi + \pi Q \ . \qquad (2.27)$$

2.7 Water vapor equation

If q is the water vapor mixing ratio, C the rate of condensation and E the rate of evaporation per unit mass of dry air, the continuity equation for water vapor can be written as

$$\frac{dq}{dt} = -C + E \ , \qquad (2.28)$$

or in flux form as

$$\frac{\partial}{\partial t}(\pi q) + \nabla_{\sigma}\cdot(\pi \mathbf{v} q) + \frac{\partial}{\partial \sigma}(\pi \dot{\sigma} q) = \pi(-C + E) \ . \qquad (2.29)$$

In cylindrical coordinates and with the axisymmetric assumption, (2.29) can be written

$$\frac{\partial}{\partial t}(\pi q) + \frac{\partial}{r\partial r}(\pi r u q) + \frac{\partial}{\partial \sigma}(\pi \dot{\sigma} q) = \pi(-C + E) \ . \qquad (2.30)$$

2.8 Summary of the complete set of equations (continuous form)

The set of model variables consists of five prognostic variables and five diagnostic variables. The five prognostic variables are π, u, v, T and q, all of which are functions of the three independent variables (r, σ, t) with the exception of π, which is a function of only (r, t). The five diagnostic variables are $\dot{\sigma}$, p, α, Φ and θ, all of which are functions of (r, σ, t).

The complete set of model equations consists of five prognostic equations and five diagnostic equations. The five prognostic equations are the vertically integrated continuity equation (2.10), the horizontal momentum equations (2.20) and (2.21), the thermodynamic equation (2.27), and the water vapor equation (2.30). The five diagnostic equations are the continuity equation (2.11), the definition of sigma (2.1), the gas law (2.22), the hydrostatic equation (2.3), and the definition of potential temperature (2.25).

The complete set of equations can be arranged for numerical integration in the following order.

$$\frac{\partial}{\partial t}(\pi r) = -\frac{\partial}{\partial r}\int_0^1 \pi r u \, d\sigma \tag{2.31}$$

$$\pi r \dot{\sigma} = -\left\{\sigma \frac{\partial}{\partial t}(\pi r) + \frac{\partial}{\partial r}\int_0^\sigma \pi r u \, d\sigma'\right\} \tag{2.32}$$

$$p = p_T + \pi\sigma \tag{2.33}$$

$$\alpha = \frac{RT}{p} \tag{2.34}$$

$$\frac{\partial \Phi}{\partial \sigma} = -\pi\alpha \ , \tag{2.35}$$

$$\frac{\partial}{\partial t}(\pi r u) = -\frac{\partial}{\partial r}(\pi r u u) - \frac{\partial}{\partial \sigma}(\pi r \dot{\sigma} u) + (f + \frac{v}{r})\pi r v$$

$$- \pi r(\frac{\partial \Phi}{\partial r} + \sigma\alpha \frac{\partial \pi}{\partial r}) + \pi r F_r \ , \tag{2.36}$$

$$\frac{\partial}{\partial t}(\pi r v) = -\frac{\partial}{\partial r}(\pi r u v) - \frac{\partial}{\partial \sigma}(\pi r \dot{\sigma} v) - (f + \frac{v}{r})\pi r u + \pi r F_\phi , \qquad (2.37)$$

$$\theta = T\left(\frac{p_o}{p}\right)^\kappa , \qquad (2.38)$$

$$\frac{\partial}{\partial t}(\pi r T) = -\frac{\partial}{\partial r}(\pi r u T) - \left(\frac{p}{p_o}\right)^\kappa \frac{\partial}{\partial \sigma}(\pi r \dot{\sigma} \theta)$$

$$+ \frac{\pi r \sigma \alpha}{c_p}\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial r}\right)\pi + \frac{\pi r Q}{c_p} , \qquad (2.39)$$

$$\frac{\partial}{\partial t}(\pi r q) = -\frac{\partial}{\partial r}(\pi r u q) - \frac{\partial}{\partial \sigma}(\pi r \dot{\sigma} q) + \pi r(-C + E). \qquad (2.40)$$

Initial conditions are required on the five prognostic variables π, u, v, T and q. The initial conditions are discussed in section 6. The procedure followed in a single prognostic cycle is as follows.

1) Calculate the tendency of π from (2.31).
2) Using the tendency of π just calculated, calculate $\pi r \dot{\sigma}$ from (2.32).
3) Using (2.33) and (2.34), calculate the geopotential Φ from (2.35).
4) Calculate the tendencies of u and v from (2.36) and (2.37).
5) Using (2.38) calculate the tendency of T from (2.39).
6) Calculate the tendency of q from (2.40).
7) Return to the first step.

3.0 PARAMETERIZATION OF CONVECTIVE SCALE PROCESSES

In this section we discuss the parameterization of cumulus convection and the parameterization of the surface exchanges of momentum and sensible and latent heat. The parameterization of the convective scale vertical transports of heat, moisture and momentum follows the theory presented by Arakawa and Schubert (1974) and Schubert (1974) and is described in 3.1. The parameterization of the surface exchanges is accomplished by bulk aerodynamic methods and is described in 3.2.

3.1 Cumulus parameterization

The cumulus parameterization theory describes the mutual interaction of a cumulus cloud ensemble with the large-scale environment. This mutual interaction is shown schematically in figure 3.1. We have conceptually grouped the equations of the theory into the three categories: feedback, static control, and dynamic control (Schubert, 1974). The feedback loop describes how the cumulus scale transport terms and source/sink terms modify the large-scale temperature and moisture fields and is discussed in section 3.1.1. The static control and dynamic control loops describe how the properties of the cloud ensemble are controlled by the large-scale fields and are discussed in sections 3.1.2 and 3.1.3 respectively. The most difficult aspect of the theory is the dynamic control, i.e. the solution of the equation for the mass flux distribution function. We have solved this equation by using optimization theory, in particular the simplex method of linear programming.

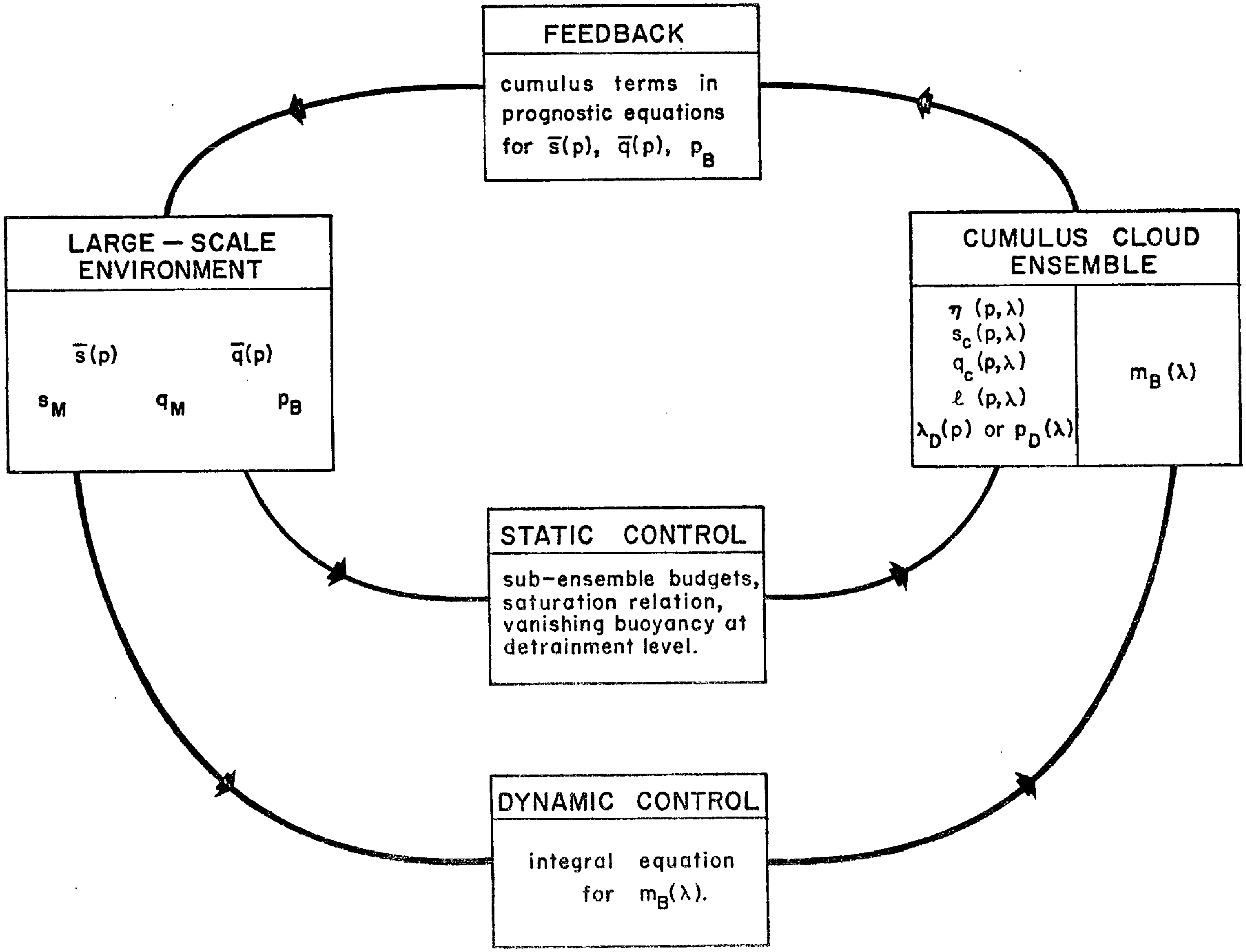

Fig. 3.1. Schematic representation of the mutual interaction of a cumulus cloud ensemble with the large-scale environment.

From a computational point of view it is convenient to formulate the dynamic control in terms of an adjustment process. Application of the simplex method to this formulation of the problem results in what we have called "the optimal adjustment method," which is discussed in 3.1.4. Although we have used the word "adjustment," our method bears very little resemblance to the moist convective adjustment methods used in many numerical models, including the GFDL and NCAR general circulation models.

3.1.1 feedback

Let us consider a horizontal area large enough to contain an ensemble of clouds but small enough so as to cover only a fraction of a large-scale disturbance. We shall refer to the vertical transports caused by motions on a scale smaller than this area as convective scale transports.

Let the large-scale environment of the cloud ensemble be divided into the subcloud mixed layer, the infinitesimally thin transition layer, and the region above (see figure 3.2). In the subcloud mixed layer the dry static energy s, water vapor mixing ratio q, and therefore the moist static energy h, are constant with height, having the respective values s_M, q_M, and h_M. The top of the subcloud mixed layer p_B is usually somewhat below cloud base p_C. Below p_B convective scale transports are accomplished by the turbulence of the mixed layer. This turbulence is confined below p_B by the stable and infinitesimally thin transition layer. Across the transition layer there can be discontinuities in

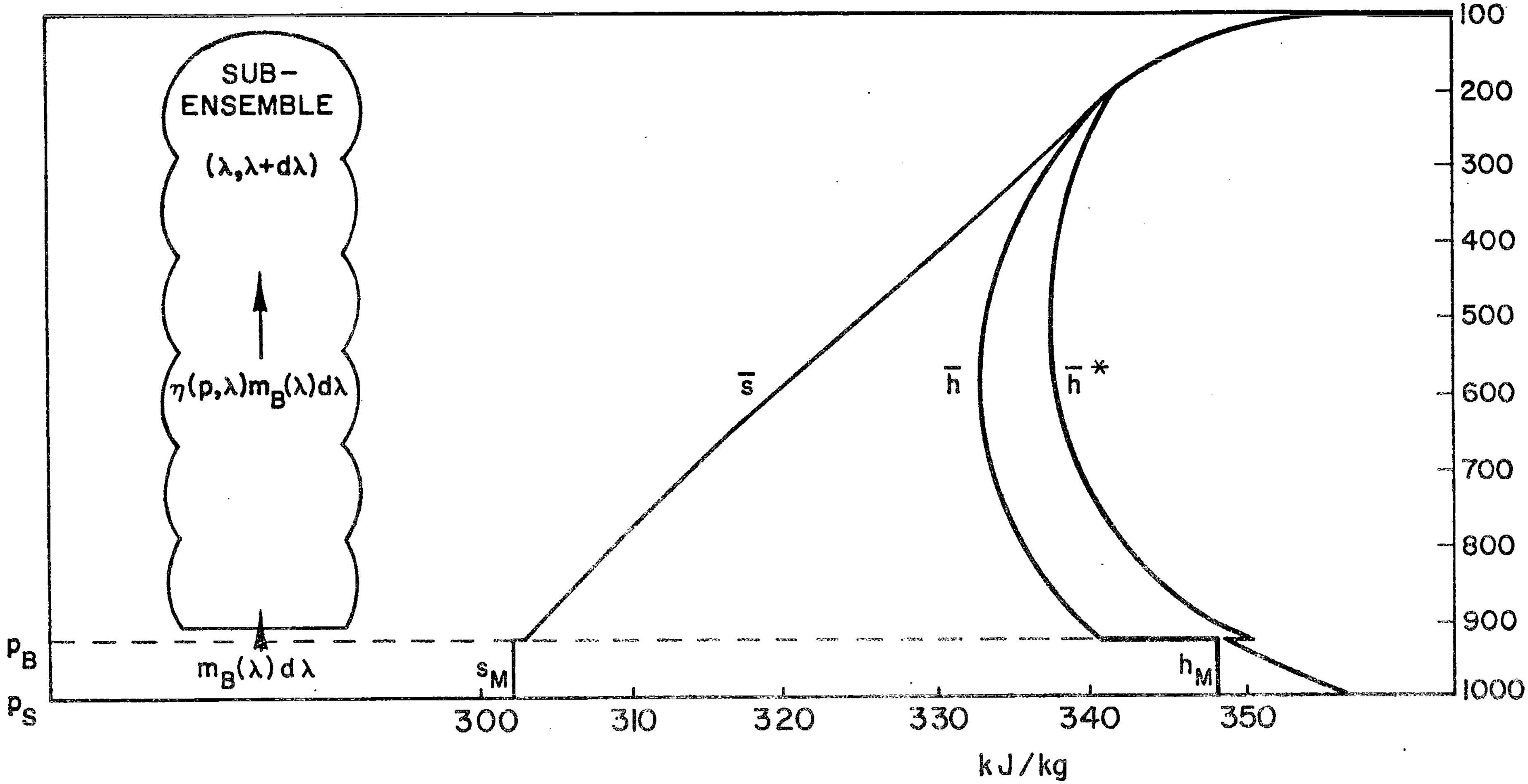

Fig. 3.2. Typical ITCZ profiles of $\bar{s}$, $\bar{h}$, and $\bar{h}^*$. Above p_B these profiles are those of Yanai, Esbensen and Chu (1973). The schematic sub-ensemble has cloud base p_c slightly above p_B. The mass flux at p is $\eta(p,\lambda)m_B(\lambda)d\lambda$, while the mass flux at p_B is $m_B(\lambda)d\lambda$.

temperature and moisture, and also discontinuities in the convective scale fluxes. Above p_B the convective scale transports are accomplished by the cloud ensemble, which is spectrally divided into "sub-ensembles" according to the fractional entrainment rate λ, small λ corresponding to deep clouds and large λ corresponding to shallow clouds.

Let us define the convective scale fluxes of dry static energy, water vapor, and liquid water as

$$F_s(p) \equiv \begin{cases} \displaystyle\int_0^{\lambda_D(p)} \eta(p,\lambda)\ [s_c(p,\lambda) - \bar{s}(p)]\ m_B(\lambda)d\lambda & p_B>p \quad (3.1a) \\ (F_s)_S + [(F_s)_B - (F_s)_S]\,\dfrac{p_S - p}{p_S - p_B} & p_S\geq p>p_B\ , \quad (3.1b) \end{cases}$$

$$F_q(p) \equiv \begin{cases} \displaystyle\int_0^{\lambda_D(p)} \eta(p,\lambda)\ [q_c(p,\lambda) - \bar{q}(p)]\ m_B(\lambda)d\lambda & p_B>p \quad (3.2a) \\ (F_q)_S + [(F_q)_B - (F_q)_S]\,\dfrac{p_S - p}{p_S - p_B} & p_S\geq p>p_B\ , \quad (3.2b) \end{cases}$$

$$F_\ell(p) \equiv \begin{cases} \displaystyle\int_0^{\lambda_D(p)} \eta(p,\lambda)\ell(p,\lambda)\ m_B(\lambda)d\lambda & p_B>p \quad (3.3a) \\ 0 & p_S\geq p>p_B\ . \quad (3.3b) \end{cases}$$

Below p_B the convective scale fluxes of s and q are linear in p with the values $(F_s)_S$ and $(F_q)_S$ at the surface p_S and the values $(F_s)_B$ and $(F_q)_B$ just below p_B. The convective scale flux of ℓ is zero everywhere below p_B.

Above p_B the convective scale fluxes are accomplished by the cloud ensemble. Let $s_c(p,\lambda)$ be the dry static energy at level p inside sub-ensemble λ and $\eta(p,\lambda)m_B(\lambda)d\lambda$ be the vertical mass flux at level p due to sub-ensemble λ . Let $\eta(p,\lambda)$ be the normalized mass flux, having the value unity at p_B. Then $m_B(\lambda)d\lambda$ is the sub-ensemble mass flux at p_B. We shall refer to $m_B(\lambda)$ as the mass flux distribution function since it gives the distribution of mass flux in λ space. The upward flux of dry static energy inside sub-ensemble λ at level p is $\eta(p,\lambda)s_c(p,\lambda)m_B(\lambda)d\lambda$. The downward flux in the environment at level p, caused by the induced subsidence of sub-ensemble λ, is given by $\eta(p,\lambda)\bar{s}(p)m_B(\lambda)d\lambda$. Thus, the total upward flux at level p due to sub-ensemble λ is $\eta(p,\lambda)[s_c(p,\lambda) - \bar{s}(p)]\, m_B(\lambda)d\lambda$. The total ensemble flux at level p is an integral over all sub-ensembles which penetrate level p. Sub-ensembles which penetrate level p have fractional entrainment rates in the interval $o\leq\lambda\leq\lambda_D(p)$, where $\lambda_D(p)$ is the fractional entrainment rate of the sub-ensemble which detrains at level p. The convective scale fluxes of water vapor and liquid water above p_B are analogous to that of dry static energy except that there is no vertical flux of liquid water in the environment since the environment contains no liquid water.

Certain combinations of the three basic fluxes given in (3.1) - (3.3) are useful. Thus, let us define the convective scale fluxes of virtual dry static energy, moist static energy, total water content, and liquid water static energy as

$$F_{sv}(p) \equiv F_s(p) + \delta\varepsilon(p)LF_q(p) \ , \tag{3.4}$$

$$F_h(p) \equiv F_s(p) + LF_q(p) \ , \tag{3.5}$$

$$F_{q+\ell}(p) \equiv F_q(p) + F_\ell(p) \ , \tag{3.6}$$

$$F_{s-L\ell}(p) \equiv F_s(p) - LF_\ell(p) \ . \tag{3.7}$$

In (3.4), $\delta = 0.608$ and $\varepsilon(p) = c_pT(p)/L$. The liquid water static energy $(s-L\ell)$ is the static energy analog of the liquid water potential temperature introduced by Betts (1973).

The governing equations for the large-scale environment are derived from the heat and moisture budgets for the region above the mixed layer, for the infinitesimally thin transition layer, and for the mixed layer. These budgets are

$$\frac{\partial \bar{s}}{\partial t} = - \bar{\mathbf{v}}\cdot\nabla\bar{s} - \bar{\omega}\frac{\partial \bar{s}}{\partial p} + g\frac{\partial}{\partial p}F_{s-L\ell} + LR + Q_R \ , \tag{3.8}$$

$$\frac{\partial \bar{q}}{\partial t} = - \mathbf{v}\cdot\nabla\bar{q} - \bar{\omega}\frac{\partial \bar{q}}{\partial p} + g\frac{\partial}{\partial p}F_{q+\ell} - R \ , \tag{3.9}$$

$$-g^{-1}\left[\frac{\partial p_B}{\partial t} + \mathbf{v}_B\cdot\nabla p_B - \bar{\omega}_B\right] = \frac{\Delta F_{s-L\ell}}{\Delta s} \ , \tag{3.10}$$

$$-g^{-1}\left[\frac{\partial p_B}{\partial t} + \mathbf{v}_B\cdot\nabla p_B - \bar{\omega}_B\right] = \frac{\Delta F_{q+\ell}}{\Delta q} \ , \tag{3.11}$$

$$\frac{\partial s_M}{\partial t} = -\mathbf{v}_M\cdot\nabla s_M + \frac{g}{p_S - p_B}\left[(F_s)_S - (F_s)_B\right] + (Q_R)_M \ , \tag{3.12}$$

$$\frac{\partial q_M}{\partial t} = -\mathbf{v}_M\cdot\nabla q_M + \frac{g}{p_S - p_B}\left[(F_q)_S - (F_q)_B\right] \ . \tag{3.13}$$

In addition to large-scale advection terms, the heat and moisture budgets above the mixed layer (equations (3.8) and (3.9)) contain convective scale flux divergence terms and the convective scale liquid water sink term R, defined by

$$R(p) \equiv g \frac{\overline{H}(p)}{p} \int_{o}^{\lambda_D(p)} \eta(p,\lambda) r(p,\lambda) m_B(\lambda) d\lambda \ . \qquad (3.14)$$

Since $\eta(p,\lambda)r(p,\lambda)m_B(\lambda)d\lambda$ is the sub-ensemble sink of liquid water, $R(p)$ is the total ensemble sink of liquid water. Q_R is the radiational heating. The detrainment forms of the heat and moisture budgets above the mixed layer were derived by Arakawa and Schubert (1974). The relation of (3.8) and (3.9) to the detrainment forms was discussed by Schubert (1974).

Since the transition layer is assumed to be infinitesimally thin, the heat and moisture budgets for this layer (equations (3.10) and (3.11)) turn out to be conditions on the discontinuities across the layer. In (3.10) and (3.11) the symbol delta represents the jump of a quantity across the transition layer, e.g., $\Delta s \equiv \bar{s}(p_B-) - s_M$ and $\Delta F_{s-L\ell} = F_{s-L\ell}(p_B-) - (F_s)_B$. The left hand side of (3.10) or (3.11) is the large-scale mass flux into the mixed layer, i.e., the large-scale mass flux relative to the moving p_B surface. Equations (3.10) and (3.11) show that discontinuities in the large-scale fluxes of s and q must be balanced by discontinuities in the convective scale fluxes of s and q.

The heat and moisture budget equations for the mixed layer (equations (3.12) and (3.13) are similar to those above the mixed layer except for the absence of vertical advection terms and precipitation terms.

$(Q_R)_M$ is the vertically averaged radiational heating of the mixed layer.

The fluxes at p_B+ can be written in terms of the surface fluxes by considering the turbulent energy balance of the mixed layer. This yields

$$(F_{sv})_B = - k(F_{sv})_S \ , \tag{3.15}$$

where k is an empirical constant with a value of approximately 0.20. Defining

$$M_B = \int_0^{\lambda_{max}} m_B(\lambda) d\lambda \ , \tag{3.16}$$

(3.8) through (3.15) can be reduced to

$$\frac{\partial \bar{s}}{\partial t} = - \ \bar{\mathbb{V}} \cdot \nabla \bar{s} - \bar{\omega} \frac{\partial \bar{s}}{\partial p} + g \frac{\partial}{\partial p} F_{s-L\ell} + LR + Q_R \ , \tag{3.17}$$

$$\frac{\partial \bar{q}}{\partial t} = - \ \bar{\mathbb{V}} \cdot \nabla \bar{q} - \bar{\omega} \frac{\partial \bar{q}}{\partial p} + g \frac{\partial}{\partial p} F_{q+\ell} - R \ , \tag{3.18}$$

$$\frac{\partial s_M}{\partial t} = - \ \mathbb{V}_M \cdot \nabla s_M + \frac{g}{p_S - p_B} \left[(F_s)_S + k \frac{\Delta s}{\Delta s_v} (F_{sv})_S \right] + (Q_R)_M \ , \tag{3.19}$$

$$\frac{\partial q_M}{\partial t} = - \ \mathbb{V}_M \cdot \nabla q_M + \frac{g}{p_S - p_B} \left[(F_q)_S + k \frac{\Delta q}{\Delta s_v} (F_{sv})_S \right] \ , \tag{3.20}$$

$$\frac{\partial p_B}{\partial t} = - \ \mathbb{V}_B \cdot \nabla p_B + \bar{\omega}_B + gM_B - \frac{gk}{\Delta s_v} (F_{sv})_S \ . \tag{3.21}$$

Thus, the temperature and moisture fields above and below p_B and the pressure at the top of the mixed layer p_B can be predicted if we can somehow determine $F_{s-L\ell}$, $F_{q+\ell}$, R, and M_B. The cumulus ensemble transport terms $F_{s-L\ell}$, $F_{q+\ell}$, M_B and the cumulus ensemble source/sink term R constitute the feedback loop shown in figure 3.1. From (3.1), (3.2), (3.3), (3.6), (3.7), (3.14) and (3.16) we can see that the determination of $F_{s-L\ell}$, $F_{q+\ell}$, R and M_B is equivalent to the determination of $\eta(p,\lambda)$, $s_c(p,\lambda)$, $q_c(p,\lambda)$, $\ell(p,\lambda)$, $r(p,\lambda)$, $\lambda_D(p)$ and $m_B(\lambda)$. All except $m_B(\lambda)$ are determined from the static control loop of the theory, which is discussed in 3.1.2. The mass flux distribution function $m_B(\lambda)$ is determined from the dynamic control loop of the theory, which is discussed in 3.1.3.

3.1.2 static control

The sub-ensemble normalized mass flux $\eta(p,\lambda)$, the sub-ensemble moist static energy $h_c(p,\lambda)$, and the sub-ensemble total water content $q_c(p,\lambda) + \ell(p,\lambda)$ are determined from the sub-ensemble mass, moist static energy, and total water budget equations. These are

$$\frac{\partial \eta(p,\lambda)}{\partial p} = - \frac{\lambda \bar{H}(p)}{p} \eta(p,\lambda) , \tag{3.22}$$

$$\frac{\partial}{\partial p} \left[\eta(p,\lambda) h_c(p,\lambda)\right] = - \frac{\lambda \bar{H}(p)}{p} \eta(p,\lambda) \bar{h}(p) \tag{3.23}$$

$$\frac{\partial}{\partial p} \left\{\eta(p,\lambda) \left[q_c(p,\lambda) + \ell(p,\lambda)\right]\right\} = - \frac{\lambda \bar{H}(p)}{p} \eta(p,\lambda) \bar{q}(p) + \frac{\bar{H}(p)}{p} \eta(p,\lambda) r(p,\lambda) , \tag{3.24}$$

where $\bar{H}$ is the scale height $R\bar{T}/g$. Between the top of the mixed layer p_B and the condensation level p_c, $q_c(p,\lambda)$ is determined from

$$\frac{\partial}{\partial p} \left[\eta(p,\lambda) q_c(p,\lambda)\right] = - \frac{\lambda \bar{H}(p)}{p} \eta(p,\lambda) \bar{q}(p) , \tag{3.25a}$$

while above p_c the air inside the clouds is saturated at a temperature only slightly different from the environment, allowing us to write

$$q_c(p,\lambda) = \bar{q}^*(p) + \frac{\gamma(p)}{1+\gamma(p)} \frac{1}{L} \left[h_c(p,\lambda) - \bar{h}^*(p)\right] , \qquad (3.25b)$$

where $\gamma \equiv \frac{L}{c_p} \left[\frac{\partial q^*}{\partial T}\right]_p$.

$\lambda_D(p)$ is given implicitly by

$$s_{vc}(p,\lambda_D(p)) = \bar{s}_v(p) , \qquad (3.26)$$

a statement that level p is a level of vanishing buoyancy (in terms of virtual dry static energy) for sub-ensemble $\lambda_D(p)$.

If $r(p,\lambda)$ is regarded as a known function of $\ell(p,\lambda)$, (3.22), (3.23), (3.24), (3.25), and (3.26) constitute a set of five equations in the five unknowns $\eta(p,\lambda)$, $s_c(p,\lambda)$, $q_c(p,\lambda)$, $\ell(p,\lambda)$, and $\lambda_D(p)$[1]. Thus, the sub-ensemble budgets (3.22) through (3.25a), the saturation relation (3.25b), and the condition of vanishing buoyancy at the detrainment level (3.26) constitute the static control loop as shown in figure 3.1.

3.1.3 dynamic control

In order to predict the large-scale fields from (3.17) through (3.21) there remains only the problem of determining $m_B(\lambda)$.

Let us define the cloud work function as

$$A(\lambda) = \int_{p_D(\lambda)}^{p_B} \eta(p,\lambda)\left[s_{vc}(p,\lambda) - \bar{s}_v(p)\right] \frac{dp}{p} . \qquad (3.27)$$

$A(\lambda)$ is an integral measure of the buoyancy force. It is also a measure

---

[1]Equations (3.22), (3.23), (3.24), and (3.25a) are differential equations which are solved from p_B upward. The boundary conditions at p_B are simply $\eta(p_B,\lambda) = 1$, $h_c(p,\lambda)^B = h_M$, $q_c(p_B,\lambda) = q_M$, and $\ell(p_B,\lambda) = 0$.

of the efficiency of kinetic energy generation for sub-ensemble λ. Since $A(\lambda)$ is actually a property of the large scale, its time derivative can be written in terms of the time derivatives of s_M, q_M, p_B, $\bar{s}(p)$, and $\bar{q}(p)$. Thus,

$$\frac{dA(\lambda)}{dt} = \frac{a(p_B,\lambda)}{\bar{H}_B}\frac{\partial s_{vM}}{\partial t} + \frac{b(p_B,\lambda)}{\bar{H}_B}\frac{\partial h_M}{\partial t}$$

$$+ \frac{\partial p_B}{\partial t}\frac{1}{p_B}\left\{[-1 + \lambda a(p_B,\lambda)]\Delta s_v + \lambda b(p_B,\lambda)\Delta h + \lambda\bar{H}_B A(\lambda)\right\}$$

$$+\int_{p_D(\lambda)}^{p_B} \eta(p,\lambda)\left\{[-1 + \lambda a(p,\lambda)]\frac{\partial \bar{s}_v(p)}{\partial t} + \lambda b(p,\lambda)\frac{\partial h(p)}{\partial t}\right\}\frac{dp}{p} \; . \quad (3.28)$$

$\bar{H}_B$ is the scale height at p_B. $a(p,\lambda)$ and $b(p,\lambda)$ are known weighting functions. The terms on the right hand side of (3.28) can be divided into two classes: those which depend on $m_B(\lambda)$ and those which do not. Thus, (3.28) can be written

$$\frac{dA(\lambda)}{dt} = \int_0^{\lambda_{max}} K(\lambda,\lambda')m_B(\lambda')d\lambda' + F(\lambda) \; , \qquad (3.29)$$

where the kernel $K(\lambda,\lambda')$ and the forcing $F(\lambda)$ are known.

The quasi-equilibrium assumption discussed by Arakawa and Schubert (1974) can be stated as an optimization problem as follows

$$\min\left\{\int_0^{\lambda_{max}} c(\lambda)\,\frac{dA(\lambda)}{dt}\,d\lambda\right\}$$

subject to

$$\frac{dA(\lambda)}{dt} = \int_0^{\lambda_{max}} K(\lambda,\lambda')m_B(\lambda')d\lambda' + F(\lambda) ,$$

$$m_B(\lambda) \geq 0 ,$$

$$\frac{dA(\lambda)}{dt} \leq 0 . \tag{3.30}$$

In (3.30) both $\frac{dA(\lambda)}{dt}$ and $m_B(\lambda)$ are regarded as unknown, while $c(\lambda)$, $K(\lambda,\lambda')$ and $F(\lambda)$ are regarded as known. Since we define the weighting function $c(\lambda)$ as negative, and since $\frac{dA(\lambda)}{dt} \leq 0$, we are attempting to minimize a non-negative quantity.

Equation (3.30) constitutes the dynamic control loop shown in figure 3.1. The parameterization theory is now closed.

3.1.4 optimal adjustment method

In a model which is discrete in time it is convenient from the computational point of view to formulate the parameterization theory in terms of an adjustment process. Thus let us divide into two parts the processes which tend to change the large-scale temperature and moisture fields--large-scale terms and convective scale terms.

$$\frac{\partial\bar{s}}{\partial t} = \left(\frac{\partial\bar{s}}{\partial t}\right)_{L.S.} + \left(\frac{\partial\bar{s}}{\partial t}\right)_{Conv.} \tag{3.31}$$

$$\frac{\partial\bar{q}}{\partial t} = \left(\frac{\partial\bar{q}}{\partial t}\right)_{L.S.} + \left(\frac{\partial\bar{q}}{\partial t}\right)_{Conv.} \tag{3.32}$$

$$\frac{\partial s_M}{\partial t} = \left(\frac{\partial s_M}{\partial t}\right)_{L.S.} + \left(\frac{\partial s_M}{\partial t}\right)_{Conv.} \tag{3.33}$$

$$\frac{\partial q_M}{\partial t} = \left(\frac{\partial q_M}{\partial t}\right)_{L.S.} + \left(\frac{\partial q_M}{\partial t}\right)_{Conv.} \tag{3.34}$$

$$\frac{\partial p_B}{\partial t} = \left(\frac{\partial p_B}{\partial t}\right)_{L.S.} + \left(\frac{\partial p_B}{\partial t}\right)_{Conv.} \tag{3.35}$$

The large-scale terms consist of radiation and large-scale horizontal and vertical advection.

$$\left(\frac{\partial \bar{s}}{\partial t}\right)_{L.S.} = - \bar{\mathbf{v}}\cdot\nabla\bar{s} - \bar{\omega}\,\frac{\partial \bar{s}}{\partial p} + Q_R \tag{3.36}$$

$$\left(\frac{\partial \bar{q}}{\partial t}\right)_{L.S.} = - \bar{\mathbf{v}}\cdot\nabla\bar{q} - \bar{\omega}\,\frac{\partial \bar{q}}{\partial p} \tag{3.37}$$

$$\left(\frac{\partial s_M}{\partial t}\right)_{L.S.} = - \mathbf{v}_M\cdot\nabla s_M + (Q_R)_M \tag{3.38}$$

$$\left(\frac{\partial q_M}{\partial t}\right)_{L.S.} = - \mathbf{v}_M\cdot\nabla q_M \tag{3.39}$$

$$\left(\frac{\partial p_B}{\partial t}\right)_{L.S.} = - \mathbf{v}_B\cdot\nabla p_B - \omega_B \tag{3.40}$$

The convective scale terms consist of convective flux divergence and source/sink terms.

$$\left(\frac{\partial \bar{s}}{\partial t}\right)_{Conv.} = g\,\frac{\partial}{\partial p}\,F_{s-L\ell} + LR \tag{3.41}$$

$$\left(\frac{\partial \bar{q}}{\partial t}\right)_{Conv.} = g\,\frac{\partial}{\partial p}\,F_{q+\ell} - R \tag{3.42}$$

$$\left(\frac{\partial s_M}{\partial t}\right)_{\text{Conv.}} = \frac{g}{p_S - p_B}\left[(F_s)_S + k\,\frac{\Delta s}{\Delta s_v}\,(F_{sv})_S\right] \tag{3.43}$$

$$\left(\frac{\partial q_M}{\partial t}\right)_{\text{Conv.}} = \frac{g}{p_S - p_B}\left[(F_q)_S + k\,\frac{\Delta q}{\Delta s_v}\,(F_{sv})_S\right] \tag{3.44}$$

$$\left(\frac{\partial p_B}{\partial t}\right)_{\text{Conv.}} = gM_B - \frac{gk}{\Delta s_v}\,(F_{sv})_S \tag{3.45}$$

In our hurricane model the large-scale terms and the convective scale terms are computed in separate subroutines which may use different time steps. Typically the large-scale terms might be computed every 0.5 minute, while the convective scale terms are computed every 2.5 minute.

Let us say that the atmosphere is stable over the depth $p_D(\lambda) \leq p \leq p_B$ if the cloud work function $A(\lambda)$, given by (3.27), is smaller than some critical value. The atmosphere is respectively neutral or unstable according to whether $A(\lambda)$ equals or exceeds the critical value.

If the large-scale terms push the atmosphere into an unstable state, it is the job of the cumulus subroutine to find a mass flux distribution function which will adjust the atmosphere back at least to (but at the same time as close as possible to) the neutral state subject to the constraint that each sub-ensemble mass flux be non-negative. This is the discrete analogue of (3.30).

Suppose we have n cloud types. Let x_i be the mass flux of the i^{th} cloud type. Let b_i be the amount that the i^{th} cloud work function exceeds the neutral value ($b_i > 0$). Cloud type j contributes an amount a_{ij} per unit mass flux to the reduction of b_i. We can now write

$$\begin{aligned} a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n &\geq b_1 \\ a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n &\geq b_2 \\ \vdots \qquad \vdots \qquad\quad & \quad \vdots \\ a_{n1} x_1 + a_{n2} x_2 + \cdots + a_{nn} x_n &\geq b_n \ , \\ x_1 \geq 0, \ x_2 \geq 0, \ \ldots, \ x_n &\geq 0 \ . \end{aligned} \tag{3.46}$$

(3.46) states that an adjustment greater than or equal to b_i must occur for each i and that each sub-ensemble mass flux must be non-negative.

The i^{th} inequality in (3.46) can be converted to an equality by introducing a surplus variable x_{n+i}, which is the surplus adjustment done to the i^{th} work function.

$$a_{i1} x_1 + a_{i2} x_2 + \cdots + a_{in} x_n - x_{n+i} = b_i \tag{3.47}$$

Note that x_{n+i} is not a mass flux but that it has dimensions of b_i. Our objective is to minimize some measure of the surplus adjustment. If we assume that this measure is linear and gross in character, we can write

$$\min \left\{ \sum_{i=n+1}^{2n} c_i x_i \right\}$$

subject to

$$
\begin{array}{llllll}
a_{11}\,x_1 + a_{12}\,x_2 + \cdots + a_{1n}\,x_n & - x_{n+1} & & & = b_1 \\
a_{21}\,x_1 + a_{22}\,x_2 + \cdots + a_{2n}\,x_n & & - x_{n+2} & & = b_2 \\
\vdots \qquad \vdots \qquad\qquad \vdots & & & & \vdots \\
a_{n1}\,x_1 + a_{n2}\,x_2 + \cdots + a_{nn}\,x_n & & & - x_{2n} & = b_n \, ,
\end{array}
$$

and $x_1 \geq 0,\ x_2 \geq 0, \ldots,\ x_{2n} \geq 0$ (3.48)

where the c_i are weights. In more compact notation we can write

$$\min cx$$

subject to

$$Ax = b$$

and

$$x \geq 0 \ . \tag{3.49}$$

We now have one minimization objective, n adjustment constraints and 2n non-negativity constraints. Solution of the problem will yield the n unknown sub-ensemble mass fluxes, the n unknown surplus adjustments, and the value of our objective function $\sum_{i=n+1}^{2n} c_i x_i$, which is a gross measure of the surplus adjustment.

The optimization problem (3.49) is easily solved by the simplex method of linear programming.

3.2 Surface flux parameterization

Surface energy interactions in the model are parameterized by the bulk aerodynamic method. The flux of sensible heat at the surface is given by (Arakawa, 1972)

$$(F_S)_S = c_p \rho_S C_D |\mathbb{V}_S| \left[T_{SEA} - T_S \right] , \tag{3.50}$$

the flux of water vapor by

$$(F_q)_S = \rho_S C_D |\mathbf{v}_S| \left[q^*(T_{SEA}, p_S) - q_S \right] , \qquad (3.51)$$

and the surface stress as

$$\boldsymbol{\tau}_S = -\rho_S C_D |\mathbf{v}_S| \mathbf{v}_S , \qquad (3.52)$$

where the drag coefficient C_D is given by

$$C_D = \begin{cases} (C_D)_{neut} \left[\dfrac{1}{1 - 7.0 \dfrac{T_{SEA} - T_S}{|\mathbf{v}_S|^2}} \right] & \text{for } T_{SEA} < T_S \\ (C_D)_{neut} \left[1 + \left(\dfrac{T_{SEA} - T_S}{|\mathbf{v}_S|^2} \right)^{\frac{1}{2}} \right] & \text{for } T_{SEA} > T_S \end{cases} \qquad (3.53)$$

and $(C_D)_{neut} = \min \left\{ \begin{matrix} 0.0025 \\ 0.0010(1 + 0.70 \, |\mathbf{v}_S|) \end{matrix} \right\}$. (3.54)

In (3.50) - (3.54) T_S, q_S and $\mathbf{v}_S$ are respectively the surface air temperature, mixing ratio and horizontal wind, T_{SEA} is the sea surface temperature, and all quantities are expressed in SI units.

4.0 FINITE DIFFERENCING

Our finite differencing schemes follow those proposed by Arakawa (1972) and Arakawa, Mintz et al. (1974) for the UCLA GCM. Our vertical differencing scheme, described in section 4.1, is identical to the UCLA scheme while our horizontal differencing scheme, described in section 4.2, is somewhat different due to the use of cylindrical coordinates and considerably simpler due to the assumption of axisymmetry. Detailed derivations of the finite difference schemes given here were first done by Arakawa (1972). We repeat some of them here for completeness and convenience of the reader. The reader wishing only a summary of the results can skip to sections 4.3 and 4.4, where we summarize the discrete form of the adiabatic frictionless part of the model.

4.1 Vertical differencing

4.1.1 vertical indexing

Let us denote by the integers k (k = 1, 2,...K) those levels at which the prognostic variables u, v, T and q are carried (see figure 4.1) and by the half integers $k + \frac{1}{2}$ (k = 0, 1, 2,...K) those levels where $\dot{\sigma}$ is carried. The upper boundary of the model corresponds to the half integer level $\frac{1}{2}$ and the lower boundary to the half integer level $K + \frac{1}{2}$. The integer level k is representative of a layer of thickness

$$\Delta\sigma_k = \sigma_{k+\frac{1}{2}} - \sigma_{k-\frac{1}{2}} , \qquad (4.1)$$

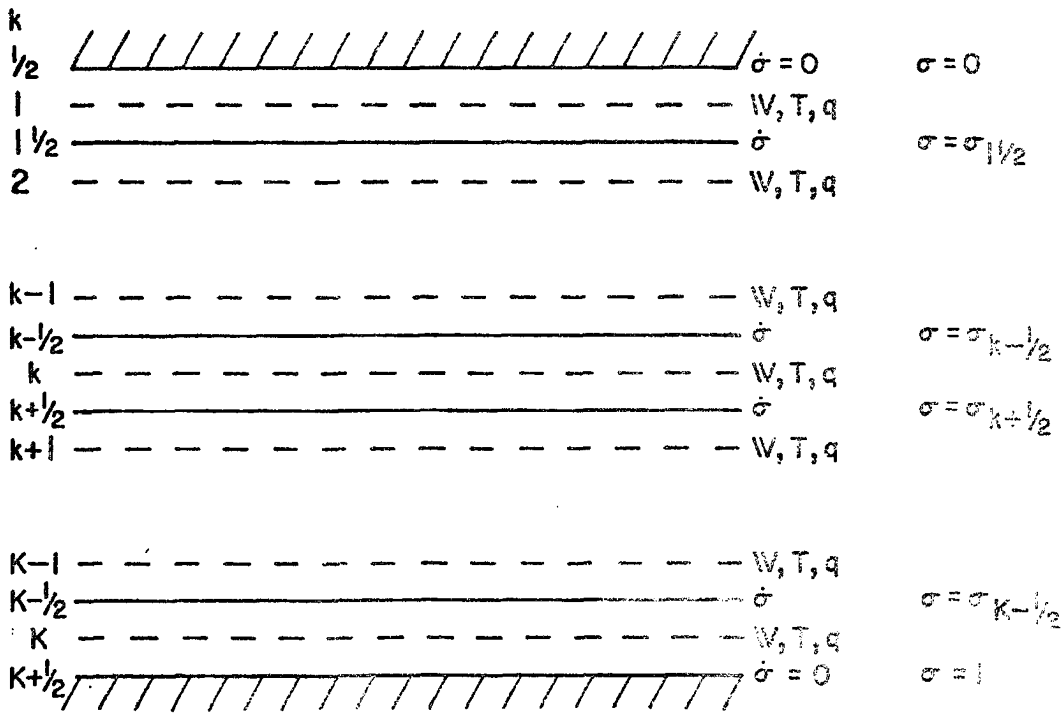

Fig. 4.1. Depiction of the vertical indexing scheme.

so that

$$\sum_{k=1}^{K} \Delta\sigma_k = 1 \ .$$

We define

$$\sigma_k \equiv \tfrac{1}{2}(\sigma_{k+\frac{1}{2}} + \sigma_{k-\frac{1}{2}}) \ . \tag{4.2}$$

4.1.2 continuity equation

The vertically discrete form of (2.8) is given by

$$\frac{\partial\pi}{\partial t} + \frac{\partial}{r\partial r}(\pi r u_k) + \frac{1}{\Delta\sigma_k}(\pi\dot{\sigma}_{k+\frac{1}{2}} - \pi\dot{\sigma}_{k-\frac{1}{2}}) = 0 \ . \tag{4.3}$$

Summing (4.3) over all k gives the vertically discrete equivalent of (2.31)

$$\frac{\partial\pi}{\partial t} = -\sum_{k=1}^{K} \frac{\partial}{r\partial r}(\pi r u_k)\Delta\sigma_k \ . \tag{4.4}$$

$(\pi\dot{\sigma})_{k+\frac{1}{2}}$ is then obtained from the vertically discrete equivalent of (2.32)

$$(\pi\dot{\sigma})_{k+\frac{1}{2}} = -\sum_{k'=1}^{k}\left[\frac{\partial\pi}{\partial t} + \frac{\partial}{r\partial r}(\pi r u_{k'})\right]\Delta\sigma_{k'} \ . \tag{4.5}$$

4.1.3 momentum equations

We write the vertically discrete momentum equations as

$$\frac{\partial}{\partial t}(\pi u_k) + \frac{\partial}{r\partial r}(\pi r u_k u_k) + \frac{1}{\Delta\sigma_k}\left[\pi\dot{\sigma}_{k+\frac{1}{2}} u_{k+\frac{1}{2}} - \pi\dot{\sigma}_{k-\frac{1}{2}} u_{k-\frac{1}{2}}\right]$$

$$- (f + \frac{v_k}{r})\pi v_k = \pi\left[\frac{\partial\Phi_k}{\partial r} + (\sigma \frac{RT}{p})_k \frac{\partial\pi}{\partial r}\right] + \pi F_{r_k} , \tag{4.6}$$

$$\frac{\partial}{\partial t}(\pi v_k) + \frac{\partial}{r\partial r}(\pi r u_k v_k) + \frac{1}{\Delta\sigma_k}\left[\pi\dot{\sigma}_{k+\frac{1}{2}} v_{k+\frac{1}{2}} - \pi\dot{\sigma}_{k-\frac{1}{2}} v_{k-\frac{1}{2}}\right]$$

$$+ (f + \frac{v_k}{r})\pi u_k = \pi F_{\phi_k} . \tag{4.7}$$

The kinetic energy equation is found by multiplying (4.6) by u_k and (4.7) by v_k and adding the result. It can then be shown that for $\pi\dot{\sigma}_{k+\frac{1}{2}}$ to remove as much kinetic energy from layer k as it adds to layer k+1, we must require that

$$u_{k+\frac{1}{2}} = \tfrac{1}{2}(u_k + u_{k+1}) , \tag{4.8}$$

$$v_{k+\frac{1}{2}} = \tfrac{1}{2}(v_k + v_{k+1}) . \tag{4.9}$$

4.1.4 first law of thermodynamics

In addition to the flux form of the thermodynamic equation given by (2.39) we have the form

$$\frac{d\theta}{dt} = \frac{1}{c_p}\frac{\theta}{T} Q . \tag{4.10}$$

The vertically discrete form of (4.10) can be written

$$\frac{\partial}{\partial t}(\pi\theta_k) + \frac{\partial}{r\partial r}(\pi r u_k \theta_k) + \frac{1}{\Delta\sigma_k}\left[\pi\dot{\sigma}_{k+\frac{1}{2}}\,\theta_{k+\frac{1}{2}} - \pi\dot{\sigma}_{k-\frac{1}{2}}\,\theta_{k-\frac{1}{2}}\right]$$

$$= \frac{1}{c_p}\left(\frac{\theta}{T}\right)_k \pi Q_k \,, \qquad (4.11)$$

where

$$\theta_k \equiv T_k \left(\frac{p_o}{p_k}\right)^{\kappa} , \qquad (4.12)$$

$$p_k = p_T + \pi\sigma_k \,. \qquad (4.13)$$

Arakawa (1972) has shown that if

$$\theta_{k+\frac{1}{2}} = \tfrac{1}{2}\,(\theta_k + \theta_{k+1}) \,, \qquad (4.14)$$

the vertical difference scheme will conserve the area integral of $\sum_{k=1}^{K} \pi\theta_k^{2}\Delta\sigma_k$ under adiabatic conditions. This is the discrete analogue of an important integral property of the continuous equation. Multiplying by $\left(\frac{p}{p_o}\right)^{\kappa}$ and rearranging allows us to write (4.11) as

$$\frac{\partial}{\partial t}(\pi c_p T_k) + \frac{\partial}{r\partial r}(\pi r u_k c_p T_k) + \left(\frac{p_k}{p_o}\right)^{\kappa} \frac{c_p}{\Delta\sigma_k}\left[\pi\dot{\sigma}_{k+\frac{1}{2}}\,\theta_{k+\frac{1}{2}} - \pi\dot{\sigma}_{k-\frac{1}{2}}\,\theta_{k-\frac{1}{2}}\right]$$

$$= \pi\left(\sigma\frac{RT}{p}\right)_k \left(\frac{\partial}{\partial t} + u_k\frac{\partial}{\partial r}\right)\pi + \pi Q_k \,. \qquad (4.15)$$

This is the vertically discrete form of (2.39).

4.1.5 total energy conservation and the hydrostatic equation

Multiplying (4.6) by u_k and (4.7) by v_k and adding the result we obtain

$$\frac{\partial}{\partial t}\left[\pi \tfrac{1}{2}(u_k{}^2 + v_k{}^2)\right] + \frac{\partial}{r\partial r}\left\{\pi r u_k \left[\tfrac{1}{2}(u_k{}^2 + v_k{}^2)\right]\right\}$$

$$+ \frac{1}{\Delta\sigma_k}\left[\pi\dot{\sigma}_{k+\frac{1}{2}}(u_k u_{k+\frac{1}{2}} + v_k v_{k+\frac{1}{2}}) - \pi\dot{\sigma}_{k-\frac{1}{2}}(u_{k-\frac{1}{2}} u_k + v_{k-\frac{1}{2}} v_k)\right]$$

$$= -\pi u_k \left[\frac{\partial \Phi_k}{\partial r} + \left(\sigma \frac{RT}{p}\right)_k \frac{\partial \pi}{\partial r}\right] + \pi u_k F_{r_k} + \pi v_k F_{\phi_k} \ . \qquad (4.16)$$

We can also show that

$$\frac{\partial}{r\partial r}(\pi r u_k \Phi_k) + \frac{1}{\Delta\sigma_k}(\pi\dot{\sigma}_{k+\frac{1}{2}}\Phi_k - \pi\dot{\sigma}_{k-\frac{1}{2}}\Phi_{k-\frac{1}{2}}) + \Phi_k \frac{\partial \pi}{\partial t}$$

$$+ \pi\left(\sigma \frac{RT}{p}\right)_k u_k \frac{\partial \pi}{\partial r} = \pi u_k \left[\frac{\partial \Phi_k}{\partial r} + \left(\sigma \frac{RT}{p}\right)_k \frac{\partial \pi}{\partial r}\right] \ . \qquad (4.17)$$

Adding (4.15), (4.16) and (4.17), multiplying the result by $\Delta\sigma_k$, summing over all k and assuming $Q = F_r = F_\phi = 0$, we obtain

$$\frac{\partial}{\partial t}\left\{\sum_{k=1}^{K} \pi\left[\tfrac{1}{2}(u_k{}^2+v_k{}^2) + c_p T_k\right]\Delta\sigma_k\right\} + \frac{\partial}{r\partial r}\left\{\sum_{k=1}^{K} \pi r u_k \left[\tfrac{1}{2}(u_k{}^2+v_k{}^2) + c_p T_k + \Phi_k\right]\Delta\sigma_k\right\}$$

$$+ \sum_{k=1}^{K}\left\{\pi\dot{\sigma}_{k+\frac{1}{2}}\left[\left(\frac{p_k}{p_o}\right)^{\kappa} c_p \theta_{k+\frac{1}{2}} + \Phi_k\right] - \pi\dot{\sigma}_{k-\frac{1}{2}}\left[\left(\frac{p_k}{p_o}\right)^{\kappa} c_p \theta_{k-\frac{1}{2}} + \Phi_k\right]\right\}$$

$$+ \frac{\partial \pi}{\partial t}\sum_{k=1}^{K}\left[\Phi_k - \pi\left(\sigma \frac{RT}{p}\right)_k\right]\Delta\sigma_k = 0 \ . \qquad (4.18)$$

This is the total energy equation for adiabatic frictionless flow. As far as the vertical differencing scheme is concerned, total energy conservation requires

$$\sum_{k=1}^{K} \left[\Phi_k - \pi\left(\sigma \frac{RT}{p}\right)_k\right]\Delta\sigma_k = 0 \tag{4.19}$$

and

$$\left(\frac{p_k}{p_o}\right)^{\kappa} c_p\theta_{k+\frac{1}{2}} + \Phi_k = \left(\frac{p_{k+1}}{p_o}\right)^{\kappa} c_p\theta_{k+\frac{1}{2}} + \Phi_{k+1} \quad \text{for} \quad k = 1,2,\ldots K\text{-}1. \tag{4.20}$$

Equation (4.19) can be rewritten as

$$\Phi_K = \sum_{k=1}^{K} \pi\left(\sigma \frac{RT}{p}\right)_k \Delta\sigma_k - \sum_{k=1}^{K-1} \sigma_{k+\frac{1}{2}} \left(\Phi_k - \Phi_{k+1}\right) , \tag{4.21}$$

while (4.20) can be rewritten as

$$\Phi_k - \Phi_{k+1} = c_p\theta_{k+\frac{1}{2}} \left[\left(\frac{p_{k+1}}{p_o}\right)^{\kappa} - \left(\frac{p_k}{p_o}\right)^{\kappa}\right] \quad \text{for } k = 1,2,\ldots K\text{-}1. \tag{4.22}$$

Using (4.22) we can rewrite (4.21) as

$$\Phi_K = \sum_{k=1}^{K} \left[\pi\sigma_k \frac{R}{p_k} \Delta\sigma_k - c_p\left(\sigma_{k+\frac{1}{2}}\, \beta_k + \sigma_{k-\frac{1}{2}}\, \alpha_k\right)\right] T_k \ , \tag{4.23}$$

and using (4.12) and (4.14) we can rewrite (4.22) as

$$\Phi_k - \Phi_{k+1} = c_p\left(\alpha_{k+1} T_{k+1} + \beta_k T_k\right) \quad \text{for} \quad k = 1,2,\ldots K\text{-}1, \tag{4.24}$$

where we have defined

$$\alpha_k \equiv \begin{cases} 0 \text{ or any value} & \text{for } k = 1 \\ \tfrac{1}{2}\left[1 - \left(\dfrac{p_{k-1}}{p_k}\right)^{\kappa}\right] & \text{for } k \geq 2 \end{cases} \tag{4.25}$$

$$\beta_k \equiv \begin{cases} \tfrac{1}{2}\left[\left(\dfrac{p_{k+1}}{p_k}\right)^{\kappa} - 1\right] & \text{for } k \leq K-1 \\ 0 & \text{for } k = K \end{cases} \tag{4.26}$$

Thus, knowing the temperature T_k we can use (4.23) to compute the geopotential of the lowermost integer level, then use (4.24) to compute the geopotential at every other integer level. Equations (4.23) and (4.24) guarantee, at least as far as the vertical differencing scheme is concerned, that total energy is conserved under adiabatic frictionless motion.

The weighting factors α_k and β_k are shown in figure 4.2 and the factor $\left[\pi\sigma_k \frac{\kappa}{p_k} \Delta\sigma_k - (\sigma_{k+\frac{1}{2}}\beta_k + \sigma_{k-\frac{1}{2}}\alpha_k)\right]$, which is the weighting factor on c_pT_k in (4.23), is given in table 4.1. In both cases π has been specified as 90 kPa, p_T as 10 kPa and $\Delta\sigma_k$ as 1/18 for all k.

4.1.6 water vapor equation

The finite difference analogue of the flux form of the water vapor equation (2.40) can be written

$$\frac{\partial}{\partial t}(\pi q_k) + \frac{\partial}{r\partial r}(\pi r u_k q_k) + \frac{1}{\Delta\sigma_k}\left[\pi\dot{\sigma}_{k+\frac{1}{2}}\, q_{k+\frac{1}{2}} - \pi\dot{\sigma}_{k-\frac{1}{2}}\, q_{k-\frac{1}{2}}\right]$$

$$= \pi(-C_k + E_k) \ . \tag{4.27}$$

Provided there are no water vapor sources or sinks (i.e. $-C_k + E_k = 0$ for all k), (4.27) insures conservation of total water vapor as far as the

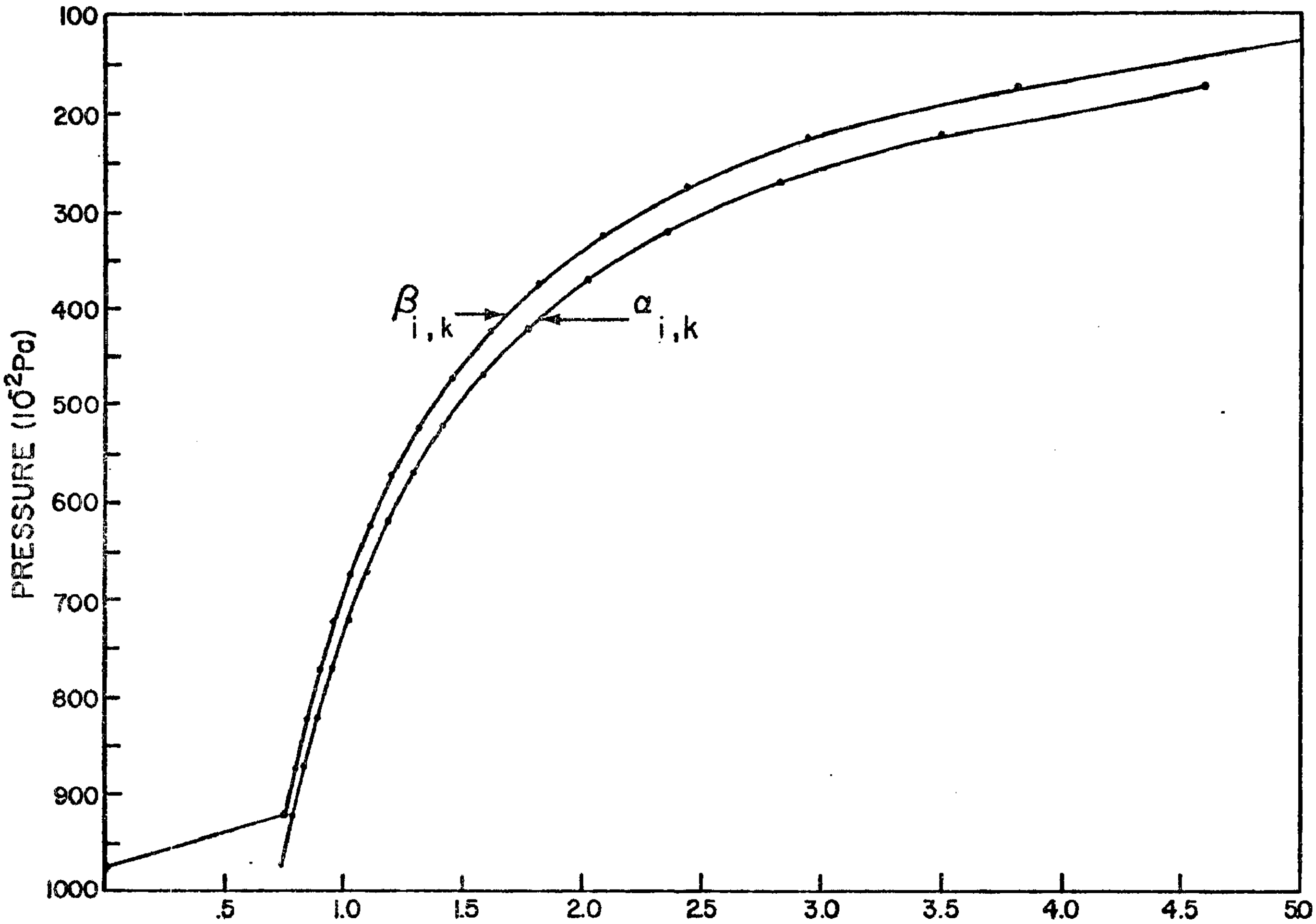

Fig. 4.2. Weighting factors $\alpha_{i,k}$ and $\beta_{i,k}$ (see equations (4.65a) and (4.65b)). π has been specified as 90 kPa, p_T as 10 kPa and $\Delta\sigma_k$ as 1/18 for all k. Units on the abscissa are 10^{-2}.

TABLE 4.1

p (units of 10^2 Pa)	$\left[\pi\sigma_k \frac{\kappa}{p_k} \Delta\sigma_k - (\sigma_{k+\frac{1}{2}} \beta_k + \sigma_{k-\frac{1}{2}} \alpha_k)\right]$ (units of 10^{-3})
125	0.37161
175	0.12082
225	0.05158
275	0.02562
325	0.01399
375	0.00813
425	0.00492
475	0.00305
525	0.00191
575	0.00118
625	0.00071
675	0.00040
725	0.00019
775	0.00005
825	-0.00005
875	-0.00012
925	-0.00016
975	7.19533

vertical differencing is concerned. Let us now concern ourselves with the problem of making a judicious choice of q at the half integer levels. We have no objective means of choosing among the many possible interpolation schemes for q at the half integer levels, but present here one possible scheme which we believe is reasonable.

Let us relate $q_{k+\frac{1}{2}}$ to q_k and q_{k+1} through an interpolation of relative humidity such that

$$\frac{q_{k+\frac{1}{2}}}{q^*_{k+\frac{1}{2}}} = \frac{1}{2}\left[\frac{q_k}{q^*_k} + \frac{q_{k+1}}{q^*_{k+1}}\right] , \qquad (4.28)$$

where

$$q^*_k = q^*(T_k,p_k) \quad \text{and} \quad q^*_{k+1} = q^*(T_{k+1},p_{k+1}) , \qquad (4.29)$$

and where $q^*_{k+\frac{1}{2}}$ is an as yet unspecified function of q^*_k and q^*_{k+1}. Equation (4.28) can also be written

$$q_{k+\frac{1}{2}} = \frac{q^*_{k+\frac{1}{2}}}{2q^*_k} q_k + \frac{q^*_{k+\frac{1}{2}}}{2q^*_{k+\frac{1}{2}}} q_{k+1} . \qquad (4.30)$$

If $q_{k+\frac{1}{2}}$ is required to lie between q_k and q_{k+1}, then

$$\frac{q^*_{k+\frac{1}{2}}}{2q^*_k} + \frac{q^*_{k+\frac{1}{2}}}{2q^*_{k+1}} = 1 , \qquad (4.31)$$

which can also be written

$$q^*_{k+\frac{1}{2}} = \frac{2q^*_k q^*_{k+1}}{q^*_k + q^*_{k+1}} . \qquad (4.32)$$

Thus, $q^*_{k+\frac{1}{2}}$ is the harmonic mean of q^*_k and q^*_{k+1}. Substituting (4.32) into (4.30) we obtain our interpolation formula for the mixing ratio at the half integer levels

$$q_{k+\frac{1}{2}} = \frac{q^*_{k+1}}{q^*_k + q^*_{k+1}} q_k + \frac{q^*_k}{q^*_k + q^*_{k+1}} q_{k+1} \ . \qquad (4.33)$$

Since q^*_{k+1} is typically greater than q^*_k, the weighting factor on q_k is typically larger than the weighting factor on q_{k+1}. Thus, $q_{k+\frac{1}{2}}$ tends to be closer to q_k than to q_{k+1}. Profiles of the weighting factors $\frac{q^*_{k+1}}{q^*_k + q^*_{k+1}}$ and $\frac{q^*_k}{q^*_k + q^*_{k+1}}$ for a mean Marshall Islands sounding with 5 kPa. resolution are shown in figure 4.3.

In the saturated case (4.33) may be a poor choice for $q_{k+\frac{1}{2}}$ and may result in what Arakawa (1972) terms "conditional instability of the computational kind" or "CICK". This type of instability can be understood by deriving the form of the thermodynamic equation which holds when large-scale condensation is occurring at level k. This equation takes the form[1]

$$c_p\left(\frac{\partial}{\partial t} + u_k \frac{\partial}{\partial r}\right)T_k = c_p\left(\frac{\partial T}{\partial p}\right)_{m,k} \sigma_k\left(\frac{\partial}{\partial t} + u_k \frac{\partial}{\partial r}\right)\pi$$

$$- \frac{1}{(1+\gamma_k)\pi\Delta\sigma_k}\left[\pi\dot{\sigma}_{k+\frac{1}{2}}(h_{k+\frac{1}{2}} - h^*_k) + \pi\dot{\sigma}_{k-\frac{1}{2}}(h^*_k - h_{k-\frac{1}{2}})\right] , \qquad (4.34)$$

where $\left(\frac{\partial T}{\partial p}\right)_{m,k}$ is the moist adiabatic lapse rate at level k, h is the moist static energy $c_pT + \Phi + Lq$ and h* is the saturation moist static energy $c_pT + \Phi + Lq^*$. If $h_{k+\frac{1}{2}} > h^*_k$, rising motion at $k+\frac{1}{2}$ (i.e. $\pi\dot{\sigma}_{k+\frac{1}{2}} < 0$) contributes to warming at level k. Similarly, if $h_{k-\frac{1}{2}} < h^*_k$, rising motion at $k-\frac{1}{2}$ (i.e. $\pi\dot{\sigma}_{k-\frac{1}{2}} < 0$) contributes to warming at level k. If $\pi\dot{\sigma}_{k+\frac{1}{2}} < 0$, there are three

[1]See Arakawa (1972), page IV-6.

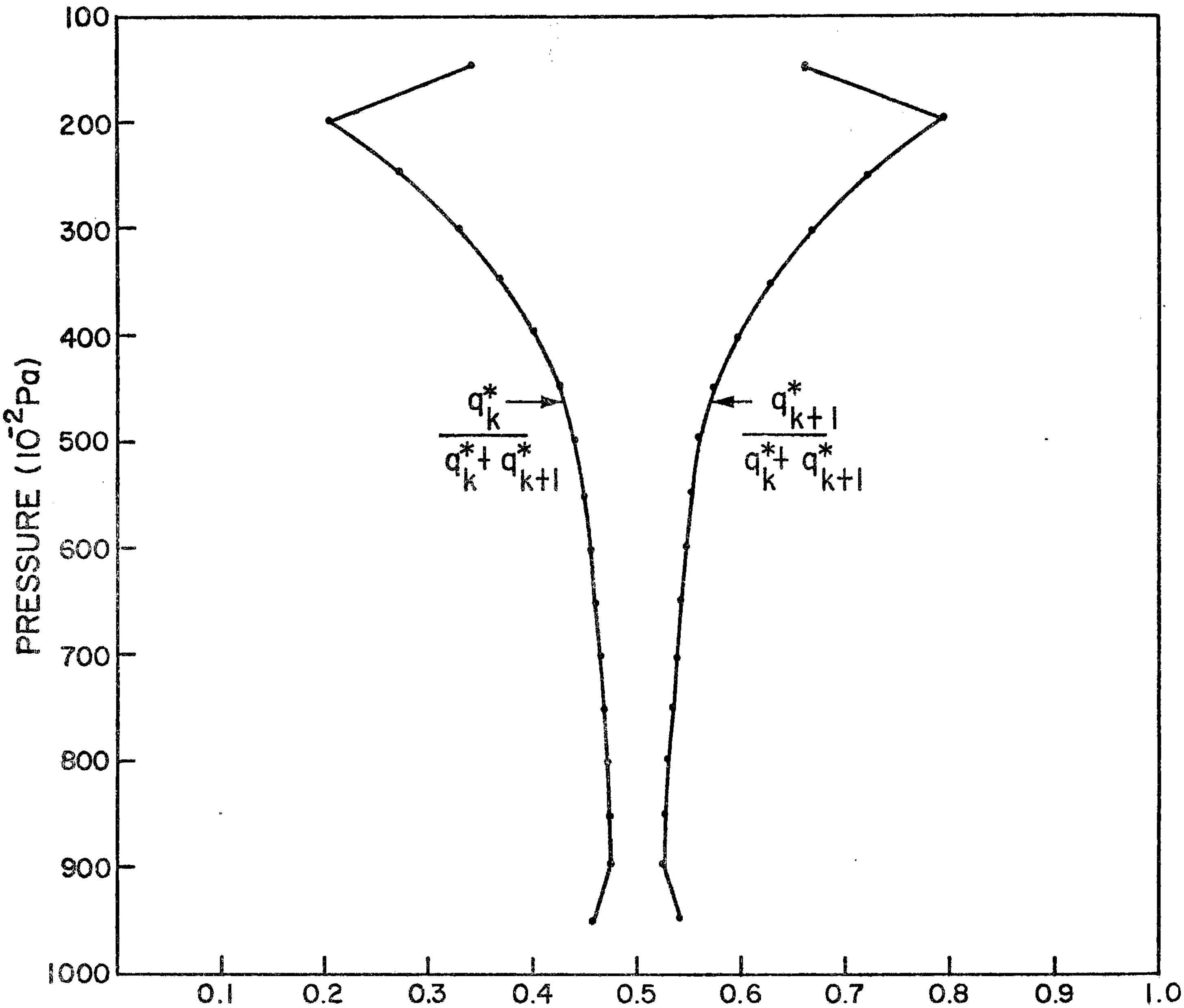

Fig. 4.3. Weighting factors $\frac{q_{k+1}^*}{q_k^* + q_{k+1}^*}$ and $\frac{q_k^*}{q_k^* + q_{k+1}^*}$ for a mean Marshall Islands sounding with 5 kPa resolution.

situations of interest at the neighboring integer levels as shown in the left column of figure 4.4. These cases are as follows:

$$\text{case (a)} \quad h_k = h_k^* \quad \text{and} \quad h_{k+1} < h_{k+1}^* \;, \tag{4.35a}$$

$$\text{case (b)} \quad h_k < h_k^* \quad \text{and} \quad h_{k+1} = h_{k+1}^* \;, \tag{4.35b}$$

$$\text{case (c)} \quad h_k = h_k^* \quad \text{and} \quad h_{k+1} = h_{k+1}^* \;. \tag{4.35c}$$

In each case one might expect that the atmosphere could be either absolutely stable (second column of figure 4.4) or conditionally unstable (third column of figure 4.4). If we require that rising motion at level $k+\frac{1}{2}$ does not contribute to warming at neighboring saturated integer levels, then we must have

$$\text{case (a)} \quad h_{k+\frac{1}{2}} \le h_k^* \;, \tag{4.36a}$$

$$\text{case (b)} \quad h_{k+1}^* \le h_{k+\frac{1}{2}} \;, \tag{4.36b}$$

$$\text{case (c)} \quad h_{k+1}^* \le h_{k+\frac{1}{2}} \le h_k^* \;. \tag{4.36c}$$

These inequalities are satisfied if $h_{k+\frac{1}{2}}$ lies on the heavy lines in the second and third columns of figure 4.4.

Let us first discuss the second column, i.e. the absolutely stable situation. Requiring that $s_{k+\frac{1}{2}}$ lie between s_k and s_{k+1}[1], and since (4.33) guarantees that $q_{k+\frac{1}{2}}$ lies between q_k and q_{k+1}, then $h_{k+\frac{1}{2}}$ must lie between h_k and h_{k+1}. Thus, it is obvious that (4.36a) and (4.36c) are satisfied. However, (4.33) may lead to an $h_{k+\frac{1}{2}}$ which does not satisfy (4.36b), i.e. (4.33) may in this case cause "conditional instability of the computational kind." If such a situation arises, we abandon (4.33) and choose $q_{k+\frac{1}{2}}$ such that $h_{k+\frac{1}{2}} = h_{k+1}^*$.

---

[1]This is analogous to (4.14).

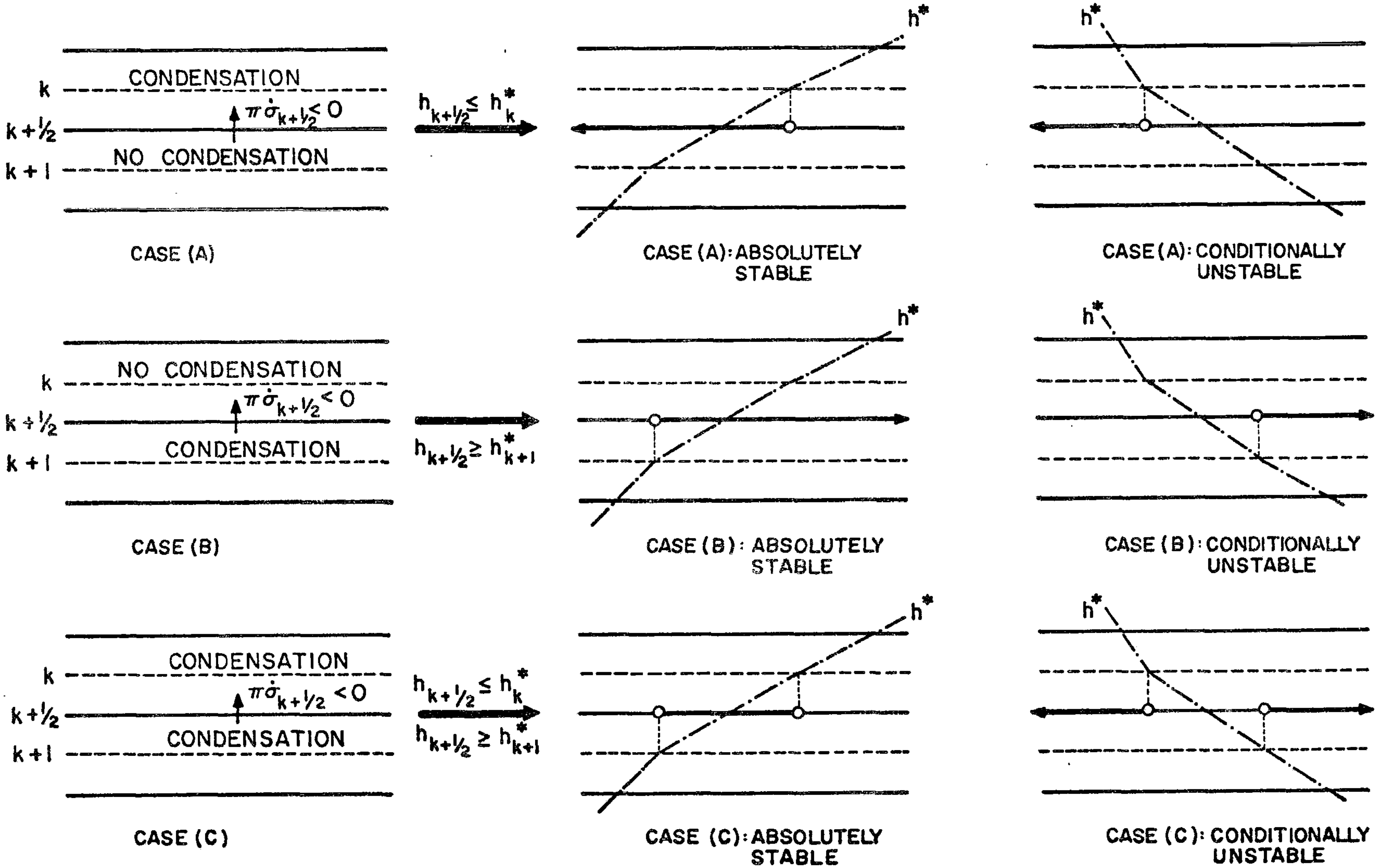

Fig. 4.4. The left column illustrates three cases in which conditional instability of the computational kind might occur. The second column portrays the absolutely stable situation and the third column the conditionally unstable situation.

Let us now discuss the third column, i.e. the conditionally unstable situation. Since $h_{k+\frac{1}{2}}$ must lie between h_k and h_{k+1} as discussed above, we can see that (4.36a) is satisfied if $h_{k+1} \leq h_k^*$. In addition, the requirement $h_{k+1} \leq h_k^*$ will eliminate case (b) and case (c). In order to maintain this requirement we resort to a moist convective adjustment scheme which is described in section 5.3. We enforce this adjustment for $1 \leq k \leq K-1$ since the cumulus parameterization theory given in section 3.1 is not general enough to include moist convection in which the updrafts originate above the mixed layer.

4.2 Horizontal differencing

4.2.1 continuity equation

We use a distribution of variables in the horizontal as shown in figure 4.5, which together with figure 4.1 leads to the depiction shown in figure 4.6. This is referred to as Scheme B by Arakawa (1972), who discusses its excellent geostrophic adjustment properties. For the equation of continuity (4.3) we use the following finite difference form

$$\frac{\partial \Pi_i}{\partial t} + F_{i+\frac{1}{2},k} - F_{i-\frac{1}{2},k} + \frac{1}{\Delta\sigma_k}\left[\dot{S}_{i,k+\frac{1}{2}} - \dot{S}_{i,k-\frac{1}{2}}\right] = 0 \quad , \qquad (4.37)$$

where we define

$$\Pi \equiv \pi r \Delta r \quad , \qquad F \equiv \pi r u \qquad \text{and} \qquad \dot{S} \equiv \Pi\dot{\sigma} \quad . \qquad (4.38)$$

The mass flux F is given as

$$F_{i+\frac{1}{2},k} = \tfrac{1}{2}\,(\pi_{i+1} + \pi_i)(ur)_{i+\frac{1}{2},k} \quad , \qquad (4.39)$$

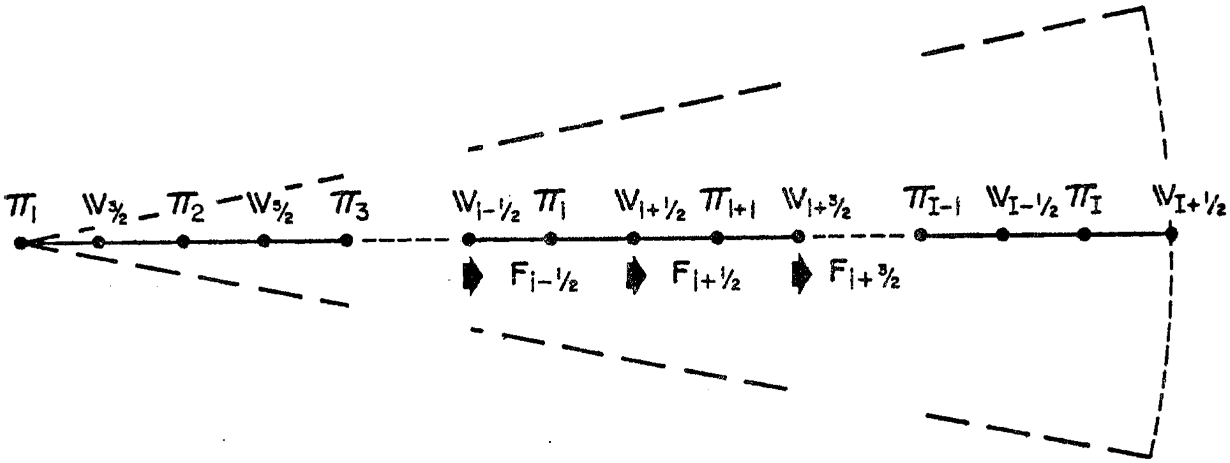

Fig. 4.5. Distribution of variables in the horizontal differencing scheme.

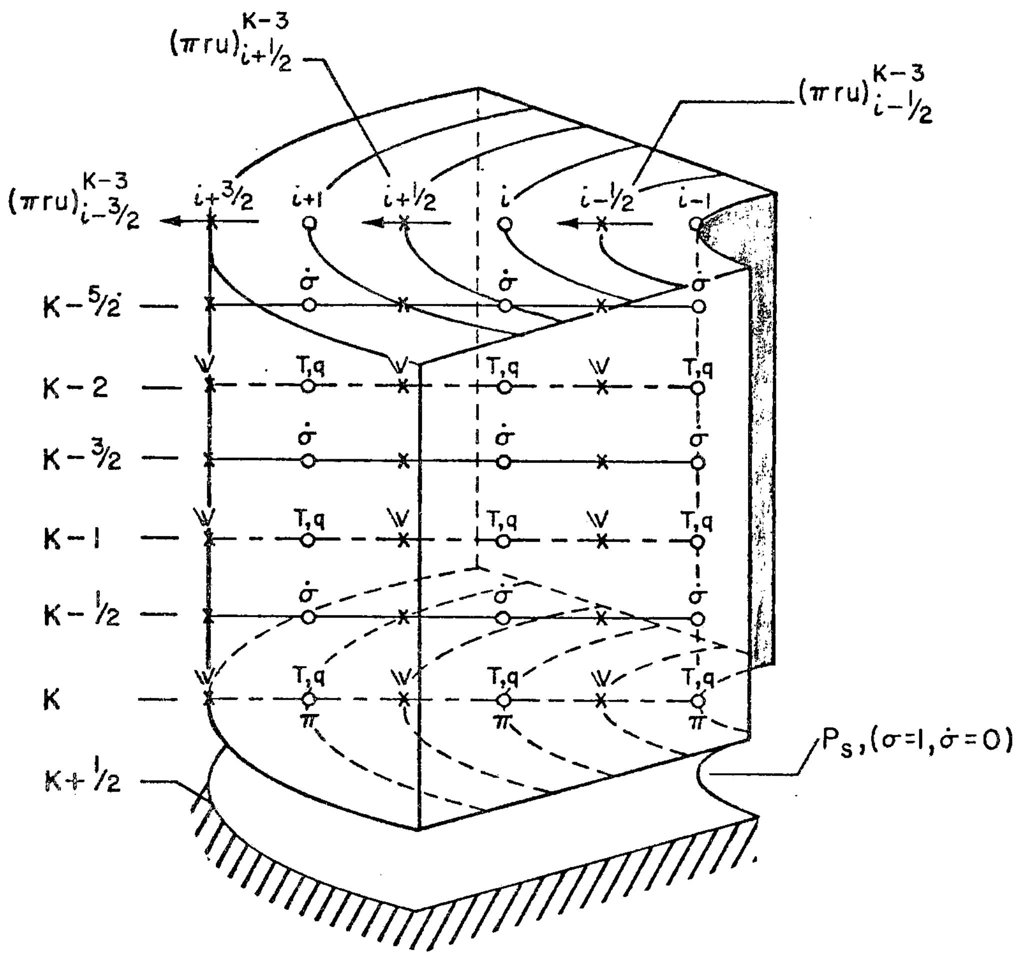

Fig. 4.6. Three dimensional depiction of the distribution and indexing schemes.

where

$$(ur)_{i+\frac{1}{2},k} = u_{i+\frac{1}{2},k}\, r_{i+\frac{1}{2}} \; . \tag{4.40}$$

4.2.2 pressure gradient force

The pressure gradient force in the radial momentum equation (2.36) can be written

$$-\pi r \frac{\partial \Phi}{\partial r} - \pi r \sigma \rho^{-1} \frac{\partial \pi}{\partial r} \; . \tag{4.41}$$

Since the first term can be rewritten as

$$-r \left[\frac{\partial}{\partial r} (\pi \Phi) - \Phi \frac{\partial \pi}{\partial r} \right] , \tag{4.42}$$

we write the discrete form as

$$-\left(\pi r \frac{\partial \Phi}{\partial r}\right)_{i+\frac{1}{2},k} = -\frac{r_{i+\frac{1}{2}}}{\Delta r} \left[\pi_{i+1}\Phi_{i+1,k} - \pi_i \Phi_{i,k} - \tfrac{1}{2}(\Phi_{i+1,k} + \Phi_{i,k})(\pi_{i+1} - \pi_i) \right] . \tag{4.43}$$

The second term in (4.41) is now written in a form consistant with that of (4.43)

$$-\left(\pi r \sigma \rho^{-1} \frac{\partial \pi}{\partial r}\right)_{i+\frac{1}{2},k} = -\frac{r_{i+\frac{1}{2}}}{2\Delta r} \left\{ \left[(\pi\sigma\rho^{-1})_{i+1,k} + (\pi\sigma\rho^{-1})_{i,k} \right] \left[\pi_{i+1} - \pi_i \right] \right\} , \tag{4.44}$$

where

$$(\pi\sigma\rho^{-1})_{i,k} = R\pi_i \sigma_{i,k} T_{i,k} / p_{i,k} \; . \tag{4.45}$$

To summarize then, the pressure force contributing to $\frac{\partial}{\partial t}(\Pi r u)_{i+\frac{1}{2},k}$ is

$$-\frac{r_{i+\frac{1}{2}}}{2} \left[(\pi_{i+1} + \pi_i)(\Phi_{i+1,k} - \Phi_{i,k}) + [(\pi\sigma\rho^{-1})_{i+1,k} + (\pi\sigma\rho^{-1})_{i,k}](\pi_{i+1} - \pi_i) \right] . \tag{4.46}$$

Although Π has been defined at the integer radial location by (4.38), Π at the u,v points (i.e. half integer radial location) has yet to be defined.

4.2.3 first law of thermodynamics

We use the form

$$\frac{\partial}{\partial t}(\Pi_i T_{i,k}) = -\tfrac{1}{2}\left[F_{i+\frac{1}{2},k}(T_{i,k}+T_{i+1,k}) - F_{i-\frac{1}{2},k}(T_{i-1,k}+T_{i,k})\right]$$
$$-\frac{1}{\Delta\sigma_k}\left(\frac{p_{i,k}}{p_o}\right)^{\kappa}\left[\dot{S}_{i,k+\frac{1}{2}}\,\theta_{i,k+\frac{1}{2}} - \dot{S}_{i,k-\frac{1}{2}}\,\theta_{i,k-\frac{1}{2}}\right]$$
$$+\frac{(\sigma\pi\rho^{-1})_{i,k}}{c_p}\left\{\frac{\partial\Pi_i}{\partial t} + \tfrac{1}{2}\left[(ru)_{i-\frac{1}{2},k}(\pi_i-\pi_{i-1}) + (ru)_{i+\frac{1}{2},k}(\pi_{i+1}-\pi_i)\right]\right\}$$
$$+\frac{\pi_i Q_i}{c_p}\quad, \tag{4.47}$$

as the horizontally discrete analogue of (4.15).

4.2.4 water vapor equation

For the horizontally discrete form of (4.27) we choose

$$\frac{\partial}{\partial t}(\Pi_i q_{i,k}) + F_{i+\frac{1}{2},k}\left(\frac{q_{i+1,k}+q_{i,k}}{2}\right) - F_{i-\frac{1}{2},k}\left(\frac{q_{i,k}+q_{i-1,k}}{2}\right)$$
$$+\frac{1}{\Delta\sigma_k}\left[\dot{S}_{i,k+\frac{1}{2}}\,q_{i,k+\frac{1}{2}} - \dot{S}_{i,k-\frac{1}{2}}\,q_{i,k-\frac{1}{2}}\right] = \Pi_i(-C+E)\quad. \tag{4.48}$$

4.2.5 momentum fluxes

Let us define the mass flux at the integer radial locations as

$$\mathcal{F}_{i,k} \equiv \tfrac{1}{2}\,(F_{i+\frac{1}{2},k} + F_{i-\frac{1}{2},k})\quad. \tag{4.49}$$

Then we may write Δr times the first three terms in (2.36) as

$$\frac{\partial}{\partial t}(\Pi_{i+\frac{1}{2}} u_{i+\frac{1}{2},k}) + \tfrac{1}{2}\left[F_{i+1,k}(u_{i+\frac{3}{2},k} + u_{i+\frac{1}{2},k}) - F_{i,k}(u_{i+\frac{1}{2},k} + u_{i-\frac{1}{2},k})\right]$$

$$+ \frac{1}{\Delta\sigma_k}\frac{1}{2}\left[\dot{S}_{i+\frac{1}{2},k+\frac{1}{2}}(u_{i+\frac{1}{2},k+1} + u_{i+\frac{1}{2},k}) - \dot{S}_{i+\frac{1}{2},k-\frac{1}{2}}(u_{i+\frac{1}{2},k} + u_{i+\frac{1}{2},k-1})\right] , \tag{4.50}$$

with a similar expression for the first three terms in (2.37). We must now determine Π and $\dot{S}$ at the half integer radial locations. Making u constant in both space and time requires that (4.50) become zero, or

$$\frac{\partial}{\partial t}\Pi_{i+\frac{1}{2}} + F_{i+1,k} - F_{i,k} + \frac{1}{\Delta\sigma_k}(\dot{S}_{i+\frac{1}{2},k+\frac{1}{2}} - \dot{S}_{i+\frac{1}{2},k-\frac{1}{2}}) = 0 \quad . \tag{4.51}$$

As pointed out by Arakawa (1972), (4.51) is necessary for the conservation of kinetic energy under a pure advective process. We can see that by defining

$$\Pi_{i+\frac{1}{2}} \equiv \frac{\Pi_i + \Pi_{i+1}}{2} \quad \text{and} \quad \dot{S}_{i+\frac{1}{2},k+\frac{1}{2}} \equiv \frac{\dot{S}_{i,k+\frac{1}{2}} + \dot{S}_{i+1,k+\frac{1}{2}}}{2} \quad , \tag{4.52}$$

(4.37) and (4.49) guarantee that (4.51) holds.

4.2.6 coriolis force

From the momentum equations (4.6) and (4.7) the coriolis force contributing to $\frac{\partial(\Pi u)}{\partial t}$ is

$$[fr\Delta r + v\Delta r]\,\pi v \quad , \tag{4.53}$$

and that contributing to $\frac{\partial(\Pi v)}{\partial t}$ is

$$-\,[fr\Delta r + v\Delta r]\,\pi u \quad . \tag{4.54}$$

We will use the form

$$\frac{1}{4}\left[(\pi_{i+1} + \pi_i)(C_{i+1,k} + C_{i,k})\right] v_{i+\frac{1}{2},k} \quad , \qquad (4.55)$$

for (4.53) and a similar form for (4.54). In (4.55)

$$C_{i,k} \equiv f r_i \Delta r + \tfrac{1}{2} (v_{i+\frac{1}{2},k} + v_{i-\frac{1}{2},k}) \Delta r \; . \qquad (4.56)$$

This form of the coriolis force allows us to maintain the relation (4.53) x u + (5.45) x v = 0 and thus avoid kinetic energy generation by the coriolis force.

4.3 Summary of the complete set of equations (differential difference form)

The finite difference analogues of (2.31) - (2.40) will now be given. We shall write the complete set in a differential difference form, leaving the time differencing scheme to section 4.4.

Our continuity equation takes the forms

$$\frac{\partial \Pi_i}{\partial t} = -\sum_{k=1}^{K} (F_{i+\frac{1}{2},k} - F_{i-\frac{1}{2},k}) \Delta\sigma_k \quad , \qquad (4.57)$$

$$\dot{S}_{i,k+\frac{1}{2}} = \dot{S}_{i,k-\frac{1}{2}} - \left(\frac{\partial \Pi_i}{\partial t} + F_{i+\frac{1}{2},k} - F_{i-\frac{1}{2},k}\right) \Delta\sigma_k \quad , \qquad (4.58)$$

where we have defined

$$\Pi_i \equiv \pi_i r_i \Delta r \quad , \qquad (4.59)$$

$$F_{i+\frac{1}{2},k} \equiv \tfrac{1}{2} (\pi_i + \pi_{i+1})(ru)_{i+\frac{1}{2},k} \quad , \qquad (4.60)$$

$$\dot{S}_{i,k+\frac{1}{2}} \equiv \Pi_i \dot{\sigma}_{i,k+\frac{1}{2}} \quad , \qquad (4.61)$$

and where

$$p_{i,k} = p_T + \pi_i \sigma_k \quad . \qquad (4.62)$$

The hydrostatic equation is given by

$$\Phi_{i,K} = \sum_{k=1}^{K}\left[\pi_i\sigma_k \frac{R}{p_{i,k}} \Delta\sigma_k - c_p(\sigma_{k-\frac{1}{2}}\alpha_{i,k} + \sigma_{k+\frac{1}{2}}\beta_{i,k})\right]T_{i,k} , \qquad (4.63)$$

and by

$$\Phi_{i,k} - \Phi_{i,k+1} = c_p\left(\alpha_{i,k+1}T_{i,k+1} + \beta_{i,k}T_{i,k}\right) , \qquad (4.64)$$

where

$$\alpha_{i,k} \equiv \begin{cases} 0 \text{ or any value} & \text{for } k = 1 \\ \frac{1}{2}\left[1 - \left(\frac{p_{i,k-1}}{p_{i,k}}\right)^{\kappa}\right] & \text{for } k \geq 2 \end{cases} \qquad (4.65a)$$

$$\beta_{i,k} \equiv \begin{cases} \frac{1}{2}\left[\left(\frac{p_{i,k+1}}{p_{i,k}}\right)^{\kappa} - 1\right] & \text{for } k \leq K-1 \\ 0 & \text{for } k = K \end{cases} . \qquad (4.65b)$$

The equation of state is

$$\rho^{-1}_{i,k} = \frac{RT_{i,k}}{p_{i,k}} . \qquad (4.66)$$

The radial and tangential momentum equations are given respectively by

$$\frac{\partial}{\partial t}(\Pi_{i+\frac{1}{2}}u_{i+\frac{1}{2},k}) = -\frac{1}{2}\left[F_{i+1,k}(u_{i+\frac{1}{2},k} + u_{i+\frac{3}{2},k}) - F_{i,k}(u_{i-\frac{1}{2},k} + u_{i+\frac{1}{2},k})\right]$$

$$- \frac{1}{\Delta\sigma_k}\frac{1}{2}\left[\dot{S}_{i+\frac{1}{2},k+\frac{1}{2}}(u_{i+\frac{1}{2},k} + u_{i+\frac{1}{2},k+1}) - \dot{S}_{i+\frac{1}{2},k-\frac{1}{2}}(u_{i+\frac{1}{2},k-1} + u_{i+\frac{1}{2},k})\right]$$

$$+ \frac{1}{4}\left[(\pi_i + \pi_{i+1})(C_{i,k} + C_{i+1,k})\right]v_{i+\frac{1}{2},k}$$

$$- \frac{r_{i+\frac{1}{2}}}{2}\left[(\pi_i+\pi_{i+1})(\Phi_{i+1,k}-\Phi_{i,k}) + \left((\sigma\pi\rho^{-1})_{i,k}+(\sigma\pi\rho^{-1})_{i+1,k}\right)(\pi_{i+1}-\pi_i)\right]$$

$$+ \Pi_{i+\frac{1}{2}}F_{r_{i+\frac{1}{2},k}} . \qquad (4.67)$$

$$\frac{\partial}{\partial t}(\Pi_{i+\frac{1}{2}}v_{i+\frac{1}{2},k}) = -\tfrac{1}{2}\left[\mathcal{F}_{i+1,k}(v_{i+\frac{1}{2},k}+v_{i+\frac{3}{2},k}) - \mathcal{F}_{i,k}(v_{i-\frac{1}{2},k}+v_{i+\frac{1}{2},k})\right]$$

$$-\frac{1}{\Delta\sigma_k}\frac{1}{2}\left[\dot{S}_{i+\frac{1}{2},k+\frac{1}{2}}(v_{i+\frac{1}{2},k}+v_{i+\frac{1}{2},k+1}) - \dot{S}_{i+\frac{1}{2},k-\frac{1}{2}}(v_{i+\frac{1}{2},k-1}+v_{i+\frac{1}{2},k})\right]$$

$$-\tfrac{1}{4}\left[(\pi_i+\pi_{i+1})(C_{i,k}+C_{i+1,k})\right]u_{i+\frac{1}{2},k} + \Pi_{i+\frac{1}{2}}F_{\phi_{i+\frac{1}{2},k}} \quad , \qquad (4.68)$$

where

$$\Pi_{i+\frac{1}{2}} \equiv \tfrac{1}{2}(\Pi_i + \Pi_{i+1}) \quad , \qquad (4.69)$$

$$\mathcal{F}_{i,k} \equiv \tfrac{1}{2}(F_{i-\frac{1}{2},k} + F_{i+\frac{1}{2},k}) \quad , \qquad (4.70)$$

$$\dot{S}_{i+\frac{1}{2},k+\frac{1}{2}} \equiv \tfrac{1}{2}(\dot{S}_{i,k+\frac{1}{2}} + \dot{S}_{i+1,k+\frac{1}{2}}) \quad , \qquad (4.71)$$

$$C_{i,k} \equiv fr_i\Delta r + \frac{\Delta r}{2}(v_{i+\frac{1}{2},k} + v_{i-\frac{1}{2},k}) \quad , \qquad (4.72)$$

$$(\sigma\pi\rho^{-1})_{i,k} \equiv \sigma_k\pi_i\frac{RT_{i,k}}{p_{i,k}} \quad . \qquad (4.73)$$

The thermodynamic equation is written in its final form as

$$\frac{\partial}{\partial t}(\Pi_i T_{i,k}) = -\tfrac{1}{2}\left[F_{i+\frac{1}{2},k}(T_{i,k}+T_{i+1,k}) - F_{i-\frac{1}{2},k}(T_{i-1,k}+T_{i,k})\right]$$

$$-\frac{1}{\Delta\sigma_k}\frac{1}{2}\left[\dot{S}_{i,k+\frac{1}{2}}(T_{i,k}+T_{i,k+1}-2\alpha_{i,k+1}T_{i,k+1}) - \dot{S}_{i,k-\frac{1}{2}}(T_{i,k}+T_{i,k-1}+2\beta_{i,k-1}T_{i,k-1})\right]$$

$$+\frac{(\sigma\pi\rho^{-1})_{i,k}}{c_p}\left\{\frac{\partial\Pi_i}{\partial t} + \tfrac{1}{2}\left[(ru)_{i-\frac{1}{2}}(\pi_i-\pi_{i-1}) + (ru)_{i+\frac{1}{2}}(\pi_{i+1}-\pi_i)\right]\right\}$$

$$+\frac{\Pi_i Q_i}{c_p} \quad . \qquad (4.74)$$

And finally, our continuity equation for water vapor may be written

$$\frac{\partial}{\partial t}(\Pi_i q_{i,k}) = -\left[F_{i+\frac{1}{2},k}\left(\frac{q_{i,k}+q_{i+1,k}}{2}\right) - F_{i-\frac{1}{2},k}\left(\frac{q_{i-1,k}+q_{i,k}}{2}\right)\right]$$

$$-\frac{1}{\Delta\sigma_k}\left[\dot{S}_{i,k+\frac{1}{2}}q_{i,k+\frac{1}{2}} - \dot{S}_{i,k-\frac{1}{2}}q_{i,k-\frac{1}{2}}\right] + \Pi_i(-C+E)_{i,k} \quad . \qquad (4.75)$$

4.4 Time differencing

The time differencing scheme for the equations of section 4.3 can be described in the following manner. Suppose we have the equation

$$\frac{d\psi}{dt} = f(\psi) \quad , \qquad (4.76)$$

where ψ is an arbitrary scaler. The leapfrog scheme is given by

$$\psi^{(n+1)} = \psi^{(n-1)} + 2\Delta t\; f\left(\psi^{(n)}\right) , \qquad (4.77)$$

and the Matsuno scheme by

$$\psi^{(n+1)*} = \psi^{(n)} + \Delta t\; f\left(\psi^{(n)}\right) , \qquad (4.78)$$

$$\psi^{(n+1)} = \psi^{(n)} + \Delta t\; f\left(\psi^{(n+1)*}\right) . \qquad (4.79)$$

The time differencing in the model is primarily handled by the leapfrog scheme. However, we periodically (every 15th step or so) employ the Matsuno scheme to prevent the divergence of the odd and even leapfrog solutions.

The use of the above explicit time differencing scheme coupled with the existence of external gravity waves requires a very small time step to satisfy the criterion for linear computational stability. For a radial grid spacing of 15 km our experience is that the time step must be less than 21 seconds. If an implicit time differencing scheme were applied to those terms which give rise to external gravity waves, a considerably larger time step could be used.

5.0 LARGE-SCALE PHASE CHANGE, DRY AND MOIST CONVECTIVE ADJUSTMENT

5.1 Large-scale phase change

The subgrid scale condensation, evaporation and precipitation caused by the parameterized cumulus convection discussed in section 4.0 can occur when the atmosphere is not saturated in a large-scale sense. In addition, large-scale condensation, evaporation and precipitation can occur when the air becomes saturated and remains saturated in a large-scale sense.

The water vapor equation can be written as

$$\frac{dq}{dt} = - C + E \tag{5.1}$$

and, neglecting for the moment heating other than that due to phase change, the first law of thermodynamics can be written as

$$\frac{d}{dt} (c_p T) - \omega\alpha = L(C - E) \ , \tag{5.2}$$

where C and E are respectively the rates of condensation and evaporation per unit mass of dry air. If the air is saturated and is remaining saturated, E vanishes and C is related to the individual time change of the saturation mixing ratio such that

$$C = - \frac{dq^*}{dt} \ . \tag{5.3}$$

In our tropical cyclone model the time step must be small enough to satisfy the linear stability criterion of Courant-Friedericks-Lewy. This criterion requires a very small time step and thus it is not necessary to compute large-scale phase change at each time step. Our procedure is to predict q and T from (5.1) and (5.2) with C and E neglected for several time steps, then check to see if q exceeds the saturation value. Thus, if in the course of integration the air becomes supersaturated on the scale of the grid, large-scale condensation and release of latent heat is assumed to occur. The excess water removed from a supersaturated layer is allowed to precipitate into the next lower layer and to evaporate completely. This process may bring that layer to supersaturation, in which case the excess is removed and precipitated to the next lower layer. When the bottom layer of the model is reached, any excess is assumed to fall to the earth's surface as large-scale precipitation.

The procedure described above is identical to that developed by A. Arakawa and J. W. Kim for the UCLA GCM. However, the computational procedure described below is somewhat different.

At level k, let the temperature be denoted by T_k and the water vapor mixing ratio by q_k. If q_k is larger than the saturation value q_k^*, a certain mass of water vapor per mass of dry air must be condensed.[1] This condensation, denoted by $C_k\Delta t$, will reduce q_k to q_k' and increase T_k to T_k',

$$q_k' = q_k - C_k\Delta t \, , \tag{5.4}$$

$$T_k' = T_k + \frac{L}{c_p} C_k\Delta t \, . \tag{5.5}$$

The new water vapor mixing ratio q_k' is the saturation value at the new temperature T_k',

---

[1] The vapor is condensed to liquid water. The ice phase is not considered.

$$q_k' = q^*(T_k', p_k) \ . \tag{5.6}$$

Equations (5.4), (5.5) and (5.6) form a closed system in the unknowns q_k', T_k' and $C_k \Delta t$. q_k' and $C_k \Delta t$ can be eliminated to give

$$q_k - q^*(T_k', p_k) - \frac{c_p}{L}(T_k' - T_k) = 0 \tag{5.7}$$

Because of the complicated form of the function $q^*(T, p)$ an explicit equation for T_k' cannot be derived from (5.7). However, an iterative scheme can be developed by applying Newton's method to (5.7). This scheme is as follows:

a) Make an initial guess of T_k for T_k', setting the iteration index, ν, to one.

b) Compute a new estimate of T_k' from the previous estimate using

$$\frac{c_p}{L}\left(T_k'^{(\nu)} - T_k'^{(\nu-1)}\right) = \frac{q_k - q^*(T_k'^{(\nu-1)}, p_k) - \frac{c_p}{L}(T_k'^{(\nu-1)} - T_k)}{1 + \gamma(T_k'^{(\nu-1)}, p_k)} \tag{5.8}$$

where $\gamma \equiv \frac{L}{c_p}\left(\frac{\partial q^*}{\partial T}\right)_p$ [1] .

c) Compute a new estimate of condensation from

$$C_k^{(\nu)} \Delta t = \frac{c_p}{L}\left[T_k'^{(\nu)} - T_k\right] . \tag{5.9}$$

---

[1] The actual expressions used for $q^*(T, p)$ and $\gamma(T, p)$ are given in Appendix A.

d) Compute a new estimate of q_k from

$$q_k^{(\nu)} = q_k - C_k^{(\nu)} \Delta t. \tag{5.10}$$

e) Test if

$$\left| \frac{q^*(T_k^{(\nu)}, p_k) - q_k^{(\nu)}}{q^*(T_k^{(\nu)}, p_k)} \right| < \text{some tolerable error} \tag{5.11}$$

If this is not true, set $\nu = \nu + 1$ and return to (b).

f) Allow the condensate $C_k \Delta t$ to fall into the next lower layer and to evaporate entirely reducing T_{k+1} by $\frac{L}{c_p} (C_k \Delta t)(\frac{\Delta\sigma_k}{\Delta\sigma_{k+1}})$, and increasing q_{k+1} by $(C_k \Delta t)(\frac{\Delta\sigma_k}{\Delta\sigma_{k+1}})$. If the bottom layer of the model is under consideration ($k = K$), the condensate reaches the earth's surface where the mass of large-scale precipitation per unit horizontal area is given by

$$\pi/g(C_K \Delta t)\Delta\sigma_K \ .$$

g) If the bottom layer of the model has not been reached, return to (a) and repeat the procedure for the next level ($k + 1$).

Our experience is that a reasonable convergence criterion is usually reached in a few iterations.

5.2 Dry convective adjustment

When potential temperature decreases with height (i.e. when $\theta_k < \theta_{k+1}$ for one or more integers $1 \leq k \leq K-1$), we assume that subgrid scale dry convection will occur and that a dry adiabatic lapse rate will result. The dry adiabatic adjustment procedure we will now describe is identical to that developed by J. W. Kim and A. Arakawa for the UCLA GCM.

At level k, let the temperature before adjustment be denoted by T_k and the temperature after adjustment by T_k'. Then, if the adjustment involves the contiguous layers beginning with k_b and ending with k_e, we can write

$$\sum_{k=k_b}^{k_e} T_k' \Delta\sigma_k = \sum_{k=k_b}^{k_e} T_k \Delta\sigma_k \quad . \tag{5.12}$$

If θ denotes the potential temperature which results from the adjustment, then

$$T_k' = \left(\frac{p_k}{p_o}\right)^{\kappa} \theta \qquad \text{for } k_b \leq k \leq k_e \quad . \tag{5.13}$$

Substituting (5.13) into (5.12) we obtain

$$\theta = \frac{\sum_{k=k_b}^{k_e} T_k \Delta\sigma_k}{\sum_{k=k_b}^{k_e} \left(\frac{p_k}{p_o}\right)^{\kappa} \Delta\sigma_k} \quad . \tag{5.14}$$

After computing θ from (5.14) we can easily compute T_k' from (5.13). This procedure may result in an unstable potential temperature stratification at neighboring intervals, in which case the procedure is repeated with new k_b and k_e. The procedure is complete when there exists no single pair of k and k+1 for which $\theta_k < \theta_{k+1}$ for any integer $1 \le k \le K-1$, i.e. when a stable temperature stratification in the entire vertical column is reached.

5.3 Moist convective adjustment

The cumulus parameterization theory given in section 3.1 is not general enough to include moist convection in which the updrafts originate above the mixed layer (i.e. altocumulus convection). As we saw in section 4.1.6, if the atmosphere above the mixed layer becomes conditionally unstable and too moist, instability can result. The instability shown in the right hand column of figure 4.4 occurs when $h_k^* < h_{k+1}^*$ and $h_{k+1} > h_k^*$ for $1 \le k \le K-2$. The important case k = K-1 is of course handled by the cumulus parameterization theory. In a model with high vertical resolution such as ours, this instability is unlikely unless the relative humidity is very large. However, if such an unstable situation arises, we must adjust h_{k+1} and h_k^* so that $h_{k+1} = h_k^*$. This is accomplished as follows.

Let us consider non-entraining clouds whose bases lie in layer k+1 and whose tops lie in layer k. These clouds produce fluxes at level $k+\frac{1}{2}$ and rain in layers k and k+1 so that the following temperature and moisture changes are produced (see (3.17) and (3.18)).

$$\frac{\Delta p_k}{g}\frac{\partial s_k}{\partial t} = \eta\left[(s_c - L\ell)_{k+\frac{1}{2}} - s_{k+\frac{1}{2}}\right] + LR_k \,, \tag{5.15}$$

$$\frac{\Delta p_{k+1}}{g}\frac{\partial s_{k+1}}{\partial t} = -\eta\left[(s_c - L\ell)_{k+\frac{1}{2}} - s_{k+\frac{1}{2}}\right] + LR_{k+1} \,, \tag{5.16}$$

$$\frac{\Delta p_k}{g}\frac{\partial q_k}{\partial t} = \eta\left[(q_c + \ell)_{k+\frac{1}{2}} - q_{k+\frac{1}{2}}\right] - R_k \ , \qquad (5.17)$$

$$\frac{\Delta p_{k+1}}{g}\frac{\partial q_{k+1}}{\partial t} = -\eta\left[(q_c + \ell)_{k+\frac{1}{2}} - q_{k+\frac{1}{2}}\right] - R_{k+1} \ . \qquad (5.18)$$

Here η represents the mass flux at level $k+\frac{1}{2}$, and R_k and R_{k+1} represent the rate of rain production in layers k and k+1 respectively. Since the clouds are non-entraining, h is constant with height in the clouds and equal to h_{k+1}. We assume that R_{k+1} is zero, i.e. that $q+\ell$ is constant with height in the clouds in layer k+1 and equal to q_{k+1}. We also assume that R_k is such that the detraining air in layer k is saturated but contains no liquid water, i.e.

$$R_k = \eta\left[(q_c + \ell)_{k+\frac{1}{2}} - (q_c)_k\right] . \qquad (5.19)$$

The rain produced over the time interval Δt is then

$$LR_k\Delta t = \eta\Delta t\left[s_k - s_{k+1} + \frac{1}{1+\gamma_k}(h_{k+1} - h_k^*)\right] . \qquad (5.20)$$

Equations (5.15) - (5.18) can now be written in discrete form as

$$\Delta s_k = \frac{g}{\Delta p_k}\left[\eta\Delta t(s_{k+1} - s_{k+\frac{1}{2}}) + LR_k\Delta t\right] , \qquad (5.21)$$

$$\Delta s_{k+1} = \frac{g}{\Delta p_{k+1}}\eta\Delta t(s_{k+\frac{1}{2}} - s_{k+1}) , \qquad (5.22)$$

$$\Delta q_k = \frac{g}{\Delta p_k}\left[\eta\Delta t(q_{k+1} - q_{k+\frac{1}{2}}) - R_k\Delta t\right] , \qquad (5.23)$$

$$\Delta q_{k+1} = \frac{g}{\Delta p_{k+1}}\eta\Delta t(q_{k+\frac{1}{2}} - q_{k+1}) . \qquad (5.24)$$

Equations (5.22) and (5.24) can be combined to give

$$\Delta h_{k+1} = \frac{g}{\Delta p_{k+1}} \eta\Delta t(h_{k+\frac{1}{2}} - h_{k+1}) \ , \tag{5.25}$$

while (5.20) and the relation $(1+\gamma_k)\Delta s_k = \Delta h_k^*$ allow (5.21) to be written

$$\Delta h_k^* = \frac{g}{\Delta p_k} \eta\Delta t\left[h_{k+1} - h_k^* + (1+\gamma_k)(s_k - s_{k+\frac{1}{2}})\right] \ . \tag{5.26}$$

Equations (5.25) and (5.26) are then combined to give

$$\Delta(h_{k+1}-h_k^*) = g\eta\Delta t\left\{\frac{1}{\Delta p_{k+1}}\left[h_{k+\frac{1}{2}}-h_{k+1}\right]- \frac{1}{\Delta p_k}\left[h_{k+1}-h_k^*+(1+\gamma_k)(s_k-s_{k+\frac{1}{2}})\right]\right\} , \tag{5.27}$$

allowing us to write the mass flux required to reduce $h_{k+1} - h_k^*$ to zero as

$$g\eta\Delta t = \frac{h_{k+1}-h_k^*}{\frac{1}{\Delta p_k}\left[h_{k+1}-h_k^*+(1+\gamma_k)(s_k-s_{k+\frac{1}{2}})\right] - \frac{1}{\Delta p_{k+1}}\left[h_{k+\frac{1}{2}}-h_{k+1}\right]} \ . \tag{5.28}$$

Our adjustment procedure is to compute $\eta\Delta t$ from (5.28), $R_k\Delta t$ from (5.20) and Δs_k, Δs_{k+1}, Δq_k and Δq_{k+1} from (5.21) - (5.24). When the adjustment procedure is applied to the pair of layers (k,k+1), we see from (5.22) that layer k+1 is warmed so that the pair of layers (k+1,k+2) must become more stable. Thus, if we start the procedure from the pair of layers (K-2,K-1) and work upwards, only one pass is required.

6.0 INITIALIZATION

To begin integration of the model initial conditions on the prognostic variables π, u, v, T and q must be specified. We shall assume that initially there is no transverse circulation, i.e. $u = 0$ everywhere, that the π, v and T fields are in gradient wind balance, and that q is a function of the vertical coordinate only and corresponds to some mean tropical sounding. Since the π, v, T and Φ fields are initially related by hydrostatic and gradient wind balance, specification of the initial v field allows computation of the initial π and T fields. Alternatively, specification of the initial π and T fields allows computation of the initial v field. Although neither of these is as straightforward as might first appear, we have chosen the first alternative as being the most convenient. Applying gradient wind balance at the sea surface we obtain

$$\left(f + \frac{v_S}{r}\right) v_S = RT_S \frac{\partial \ell n p_S}{\partial r} \quad \text{at } \sigma = 1 , \tag{6.1}$$

where $\frac{\partial \Phi}{\partial r}$ disappears since Φ is zero in our model everywhere along the $\sigma = 1$ surface. Knowing v_S and T_S everywhere, (6.1) allows us to compute p_S, and hence π, provided we specify an outer boundary condition on p_S.

The gradient wind equation at an interior point takes the form

$$\left(f + \frac{v}{r}\right) v = \frac{\partial \Phi}{\partial r} + \sigma\alpha \frac{\partial \pi}{\partial r} . \tag{6.2}$$

Differentiating (6.2) with respect to sigma and using the hydrostatic equation, we obtain

$$\sigma \frac{\partial \pi}{\partial r} \frac{\partial \alpha}{\partial \sigma} - \pi \frac{\partial \alpha}{\partial r} = \frac{\partial}{\partial \sigma} \left[f + \frac{v}{r} v \right] . \tag{6.3}$$

Since we know v and π we can regard (6.3) as a first order partial differential equation in α. Knowledge of α at the sea surface and at the outer boundary allows us to solve for α at all interior points, after which T can be determined from the equation of state.

Let us now consider the initialization procedure in discrete form. The discrete analogue of (6.1) can be written

$$\ell n p_{S_i} = \ell n p_{S_{i+1}} - \frac{(C_{i,K+\frac{1}{2}} + C_{i+1,K+\frac{1}{2}}) v_{i+\frac{1}{2},K+\frac{1}{2}}}{2r_{i+\frac{1}{2}} RT_S} , \tag{6.4}$$

allowing us to determine the surface pressure inward from the outer boundary of the model.

The discrete analogue of (6.2) can be written

$$\frac{1}{2r_{i+\frac{1}{2}}} (C_{i,k} + C_{i+1,k}) v_{i+\frac{1}{2},k}$$

$$= \Phi_{i+1,k} - \Phi_{i,k} + \frac{\pi_{i+1} - \pi_i}{\pi_i + \pi_{i+1}} R \left(\sigma_k \pi_i \frac{T_{i,k}}{p_{i,k}} + \sigma_k \pi_{i+1} \frac{T_{i+1,k}}{p_{i+1,k}} \right) . \tag{6.5}$$

Taking the difference of (6.5) applied at k and k + 1, and using the form of the hydrostatic equation given by[1]

$$\Phi_{i,k} - \Phi_{i,k+1} = c_p (\alpha_{i,k+1} T_{i,k+1} + \beta_{i,k} T_{i,k}) , \tag{6.6}$$

---

[1] In (6.6), $\alpha_{i,k}$ and $\beta_{i,k}$ are given by (4.65).

we obtain

$$c_p(\alpha_{i,k+1}\, T_{i,k+1} + \beta_{i,k}\, T_{i,k}) + \frac{\pi_{i+1}-\pi_i}{\pi_i+\pi_{i+1}}\, \pi_i R \left[\sigma_{k+1}\, \frac{T_{i,k+1}}{p_{i,k+1}} - \sigma_k\, \frac{T_{i,k}}{p_{i,k}}\right]$$

$$= \Phi_{i+1,k} - \Phi_{i+1,k+1} + \frac{\pi_{i+1}-\pi_i}{\pi_i+\pi_{i+1}}\, \pi_{i+1} R \left[\sigma_k\, \frac{T_{i+1,k}}{p_{i+1,k}} - \sigma_{k+1}\, \frac{T_{i+1,k+1}}{p_{i+1,k+1}}\right]$$

$$+ \frac{1}{2r_{i+\frac{1}{2}}} \left[(C_{i,k+1} + C_{i+1,k+1}) v_{i+\frac{1}{2},k+1} - (C_{i,k} + C_{i+1,k}) v_{i+\frac{1}{2},k}\right]. \quad (6.7)$$

The gradient wind balance in the lowest layer is

$$\Phi_{i,K} - \frac{\pi_{i+1}-\pi_i}{\pi_i+\pi_{i+1}}\, R \left(\sigma_K \pi_i\, \frac{T_{i,K}}{p_{i,K}} + \sigma_K \pi_{i+1}\, \frac{T_{i+1,K}}{p_{i+1,K}}\right)$$

$$= \Phi_{i+1,K} - \frac{1}{2r_{i+\frac{1}{2}}} (C_{i,K} + C_{i+1,K}) v_{i+\frac{1}{2},K} \;, \quad (6.8)$$

where $\Phi_{i,K}$ is given by the vertically integrated form of the hydrostatic equation

$$\Phi_{i,K} = \sum_{k=1}^{K} \left[\pi_i \sigma_k\, \frac{R}{p_{i,k}}\, \Delta\sigma_k - c_p(\sigma_{k-\frac{1}{2}}\, \alpha_k + \sigma_{k+\frac{1}{2}}\, \beta_k)\right] T_{i,k} \;. \quad (6.9)$$

Equations (6.7) and (6.8) can thus be written

$$\left[c_p\beta_{i,k} - \frac{\pi_{i+1}-\pi_i}{\pi_i+\pi_{i+1}}\,\pi_i R\,\frac{\sigma_k}{p_{i,k}}\right]T_{i,k} + \left[c_p\alpha_{i,k+1} + \frac{\pi_{i+1}-\pi_i}{\pi_i+\pi_{i+1}}\,\pi_i R\,\frac{\sigma_{k+1}}{p_{i,k+1}}\right]T_{i,k+1}$$

$$= \Phi_{i+1,k} - \Phi_{i+1,k+1} + \frac{\pi_{i+1}-\pi_i}{\pi_i+\pi_{i+1}}\,\pi_{i+1}R\left[\sigma_k\,\frac{T_{i+1,k}}{p_{i+1,k}} - \sigma_{k+1}\,\frac{T_{i+1,k+1}}{p_{i+1,k+1}}\right]$$

$$+ \frac{1}{2r_{i+\frac{1}{2}}}\left[(C_{i,k+1} + C_{i+1,k+1})v_{i+\frac{1}{2},k+1} - (C_{i,k} + C_{i+1,k})v_{i+\frac{1}{2},k}\right] \quad (6.10)$$

$$k = 1,2,\ldots\ldots\ldots K-1,$$

$$\sum_{k=1}^{K}\left[\pi_i\sigma_k\,\frac{R}{p_{i,k}}\,\Delta\sigma_k - c_p(\sigma_{k-\frac{1}{2}}\,\alpha_k + \sigma_{k+\frac{1}{2}}\,\beta_k)\right]T_{i,k} - \frac{\pi_{i+1}-\pi_i}{\pi_i+\pi_{i+1}}\,R\sigma_K\pi_i\,\frac{T_{i,K}}{p_{i,K}}$$

$$= \Phi_{i+1,K} + \frac{\pi_{i+1}-\pi_i}{\pi_i+\pi_{i+1}}\,R\sigma_K\pi_{i+1}\,\frac{T_{i+1,K}}{p_{i+1,K}} - \frac{1}{2r_{i+\frac{1}{2}}}\,(C_{i,K} + C_{i+1,K})v_{i+\frac{1}{2},K}. \quad (6.11)$$

Knowing $T_{i+1,k}$ and $\Phi_{i+1,k}$ (k=1,2...K) we can solve (6.10) and (6.11) as a linear system of K equations for $T_{i,k}$ (k=1,2,...K). We then use (6.9) to compute $\Phi_{i,K}$ and (6.6) to compute $\Phi_{i,k}$ (k=K-1,K-2,...1). This procedure is repeated for each i until we reach i=1. To initiate the procedure we assume some vertical temperature profile at the outer boundary (e.g. a mean tropical sounding) and compute the resulting profile of Φ from (6.9) and (6.6).

Thus, with the specification of u, v and q and the determination of π and T from hydrostatic and gradient wind balance, the initialization is complete.

7.0 CONCLUSIONS

We have presented the detailed design of an axisymmetric tropical cyclone model. The model is based on the primitive equations in the sigma coordinate system. The cumulus parameterization used in the model follows the theory presented by Arakawa and Schubert (1974). The finite differencing schemes follow those developed by Arakawa for the UCLA GCM. Further discussion of and experiments with the parameterization theory will be given by P. Silva Dias and W. Schubert.[1] In addition, results of numerical integrations with the tropical cyclone model will be given by J. Hack and W. Schubert.[1]

In the present model there are two shortcomings which we are attempting to correct.

The first shortcoming of the present model is that virtual temperature effects are not incorporated into the large-scale equations. The incorporation of virtual temperature effects is not as straightforward as one might think. This situation results from the fact that water vapor effects must appear not only in $\nabla\Phi$ but also in $\nabla\pi$.

The second shortcoming is that a mixed layer of variable depth is not included in the present model. If a mixed layer of variable depth is introduced into the present sigma coordinate model, certain computational disadvantages arise because the top of the mixed layer is not necessarily a coordinate surface. However, it is possible to design a generalized sigma coordinate system in which both the ground and the top of the mixed layer are coordinate surfaces. Such a generalized sigma coordinate system is presently being incorporated into the model.

[1]These reports will appear in this Atmospheric Science series.

ACKNOWLEDGEMENTS

We have benefited from discussions with many colleagues including Drs. Akio Arakawa, William Gray, Katsuyuki Ooyama, Stanley Rosenthal, Messrs. William Fingerhut, William Frank, and Pedro Silva Dias. We are also indebted to Dorothy Chapman for her help in preparing this report.

The research reported here was sponsored by the Global Atmospheric Research Program, Climate Dynamics Research Section, Grants DES 74-11438, ATM 76-09370 and OCD 74-21678.

REFERENCES

Anthes, R. A., 1972: Development of asymmetries in a three-dimensional numerical model of the tropical cyclone. Mon. Wea. Rev., 100, 461-476.

_______, S. L. Rosenthal and J. W. Trout, 1971a: Preliminary results from an asymmetric model of the tropical cyclone. Mon. Wea. Rev., 99, 744-758.

_______, J. W. Trout and S. L. Rosenthal, 1971b: Comparisons of tropical cyclone simulations with and without the assumption of circular symmetry. Mon. Wea. Rev., 99, 759-766.

Arakawa, A., 1972: Design of the UCLA general circulation model. Numerical simulation of weather and climate, Technical Report 7, Dept. of Meteorology, University of California, Los Angeles.

_______, Y. Mintz, and collaborators, 1974: The UCLA atmospheric general circulation model. Dept. of Meteorology, UCLA, Los Angeles, California.

_______, and W. H. Schubert, 1974: Interaction of a cumulus cloud ensemble with the large-scale environment, Part I. J. Atmos. Sci., 31, 674-701.

Betts, A. K., 1973: Nonprecipitating cumulus convection and its parameterization. Quart. J. Roy. Meteor. Soc., 99, 178-196.

Charney, J. G., and A. Eliassen, 1964: On the growth of the hurricane depression. J. Atmos. Sci., 21, 68-74.

Kuo, H. L., 1965: On the formation and intensification of tropical cyclones through latent heat release by cumulus convection. J. Atmos. Sci., 22, 40-63.

Kurihara, Y., 1975: Budget analysis of a tropical cyclone simulated in an axisymmetric numerical model. J. Atmos. Sci., 32, 25-59.

_______, and R. E. Tuleya, 1974: Structure of a tropical cyclone developed in a three-dimensional numerical simulation model. J. Atmos. Sci., 31, 893-919.

Madala, R. V., and S. A. Piacsek, 1975: Numerical simulation of asymmetric hurricanes on a β-plane with vertical shear. Tellus, 27, 453-468.

Mathur, M. B., 1972: Simulation of an asymmetric hurricane with a fine mesh multiple grid primitive equation model. Rept. No. 72-1, Dept. of Meteorology, Florida State University.

Ogura, Y., 1964: Frictionally controlled, thermally driven circulations in a circular vortex with application to tropical cyclones. J. Atmos. Sci., 21, 610-621.

Ooyama, K., 1964: A dynamical model for the study of tropical cyclone development. Geofisica Internacional, 4, 187-198.

_______, 1969a: Numerical simulation of the life cycle of tropical cyclones. J. Atmos. Sci., 26, 3-40.

_______, 1969b: Numerical simulation of tropical cyclones with an axi-symmetric model. Proc. WMO/IUGG Symposium Numerical Weather Prediction, Tokyo, 26 November-4 December 1968, Japan Meteor. Agency, 81-88.

_______, 1971: A theory on parameterization of cumulus convection. J. Meteor. Soc. Japan, 46, 178-201.

Peng, L., and H. L. Kuo, 1975: A numerical simulation of the development of tropical cyclones. Tellus, 27, 133-144.

Phillips, N. A., 1957: A coordinate system having some special advantages for numerical forecasting. J. Meteor., 14, 184-185.

Rosenthal, S. L., 1969a: Preliminary results from numerical experiments with a primitive equation model designed to simulate the development of tropical cyclones. Proc. WMO/IUGG Symposium Numerical Weather Prediction, Tokyo, 26 November-4 December 1968, Japan Meteor. Agency, 49-59.

_______, 1969b: Numerical experiments with a multilevel primitive equation model designed to simulate the development of tropical cyclones, experiment I. ESSA Technical Memorandum ERLTM-NHRL 82, U. S. Department of Commerce, National Hurricane Research Laboratory, Miami, Florida, 36 pp.

_______, 1970: A circularly symmetric primitive equation model of tropical cyclone development containing an explicit water vapor cycle. Mon. Wea. Rev., 98, 643-663.

_______, 1971: The response of a tropical cyclone model to variations in boundary layer parameters, initial conditions, lateral boundary conditions, and domain size. Mon. Wea. Rev., 99, 767-777.

_______, 1973: Hurricane modeling experiments with a new parameterization for cumulus convection. NOAA Technical Memorandum, ERL WMPO-4, U. S. Department of Commerce, National Hurricane Research Laboratory, Miami, Florida, 41 pp.

Schubert, W. H., 1974: Cumulus parameterization theory in terms of feedback and control. Colorado State University Atmospheric Science Paper No. 226.

Sundqvist, H., 1970a: Numerical simulation of the development of tropical cyclones with a ten-level model. Part I. Tellus, 22, 359-390.

_______, 1970b: Numerical simulation of the development of tropical cyclones with a ten-level model. Part II. Tellus, 22, 504-510.

_______, 1972: Mean tropical storm behavior in experiments related to modification attempts. Tellus, 24, 6-12.

Tuleya, R. E., and Y. Kurihara, 1975: The energy and angular momentum budgets of the three-dimensional tropical cyclone model. J. Atmos. Sci., 32, 287-301.

Yamasaki, M., 1968a: Numerical simulation of tropical cyclone development with the use of primitive equations. J. Meteor. Soc. Japan, 46, 178-201.

_______, 1968b: A tropical cyclone model with parameterized vertical partition of released latent heat. J. Meteor. Soc. Japan, 46, 202-213.

_______, 1968c: Detailed analysis of a tropical cyclone simulated with a 13-layer model. Papers Meteor. Geophys., 19, 559-585.

Yanai, M., S. Ebensen, and J.-H. Chu, 1973: Determination of bulk properties of tropical cloud clusters from large-scale heat and moisture budgets. J. Atmos. Sci., 30, 611-627.

APPENDIX

DETERMINATION OF SATURATION MIXING RATIO AND GAMMA

The saturation mixing ratio q* is given by

$$q^*(T,p) = 0.622 \frac{e_s(T)}{p - e_s(T)} \quad , \qquad \text{(A1)}$$

where $e_s(T)$ is the saturation vapor pressure over a plane surface of water. Teten's formula for $e_s(T)$ is

$$e_s(T) = 610.78 \exp\left[\frac{a(T - 273.16)}{T - b}\right] \quad , \qquad \text{(A2)}$$

where $e_s(T)$ is in Pa, T is in degrees Kelvin, a = 17.269, and b = 35.86.

The expression for the function γ can be obtained from (A1) and (A2) and is given by

$$\gamma(T,p) = \frac{L}{c_p}\left(\frac{\partial q^*}{\partial T}\right)_p = q^*(T,p)\left[\frac{p}{p - e_s(T)}\right] \frac{\partial \ell n\, e_s(T)}{\partial T} \quad ,$$

which can also be written

$$\gamma(T,p) = 4098.03 \frac{L}{c_p} \frac{p}{p - e_s(T)} \frac{q^*(T,p)}{(T - b)^2} \quad . \qquad \text{(A3)}$$